Compressive sensing in the analog world

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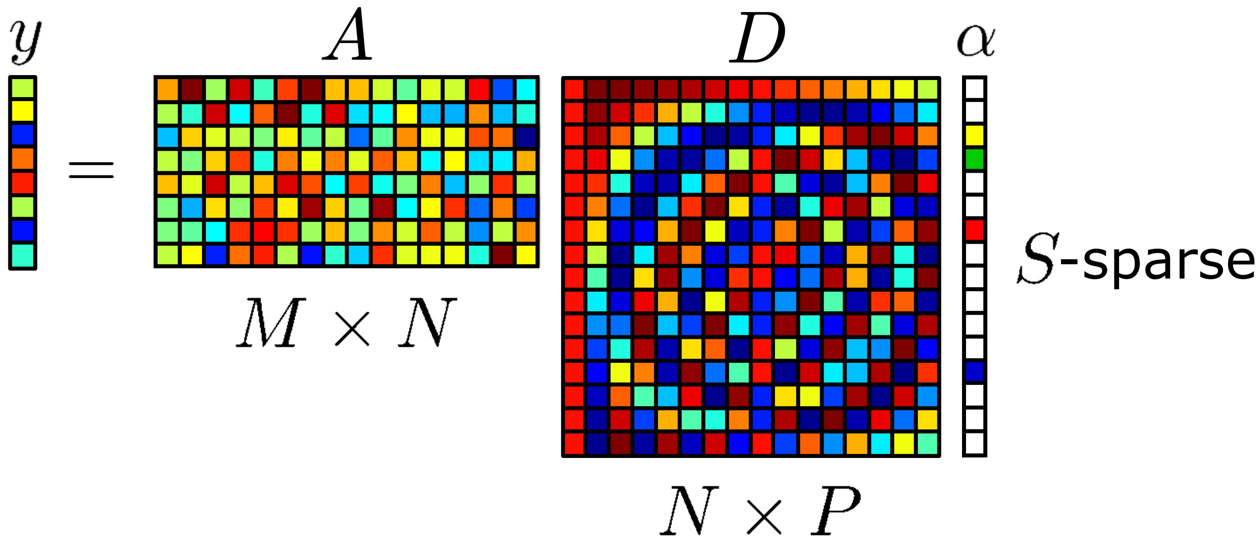
Compressive Sensing

$$y = Ax \qquad x = D\alpha$$



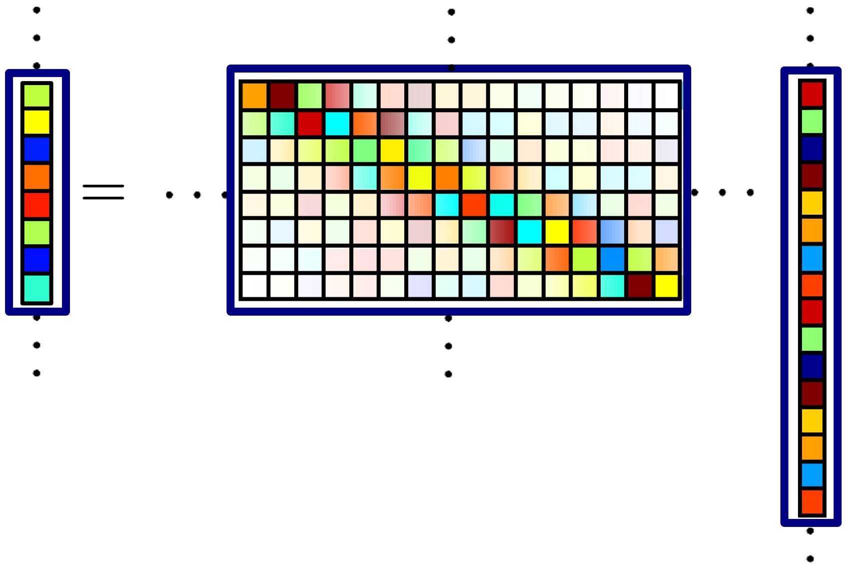
Can we really acquire analog signals with "CS"?

Challenge 1

Map analog sensing to matrix multiplication

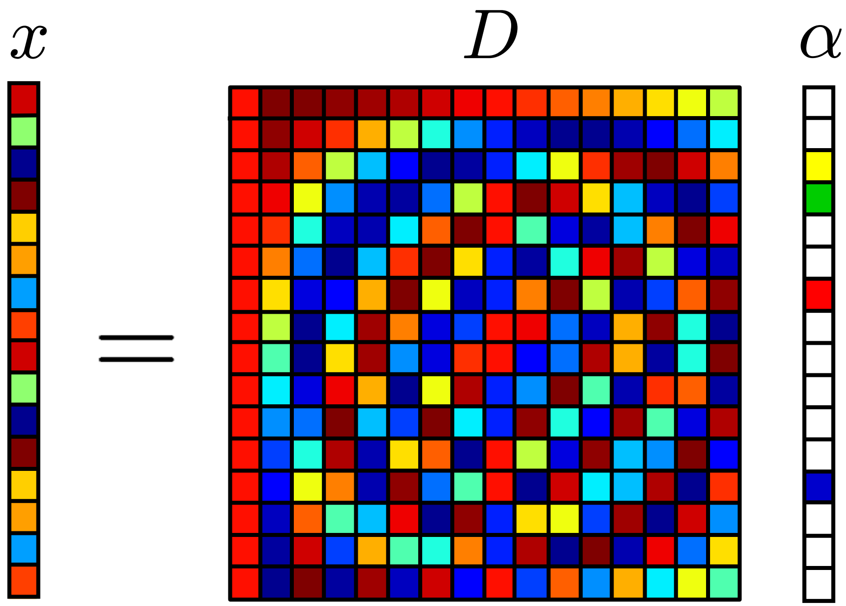
If $x(t)$ is bandlimited, $x(t) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}(t/T_s - n)$

$$y[m] = \langle \phi_m(t), x(t) \rangle = \sum_{n=-\infty}^{\infty} x[n] \langle \phi_m(t), \mathrm{sinc}(t/T_s - n) \rangle$$



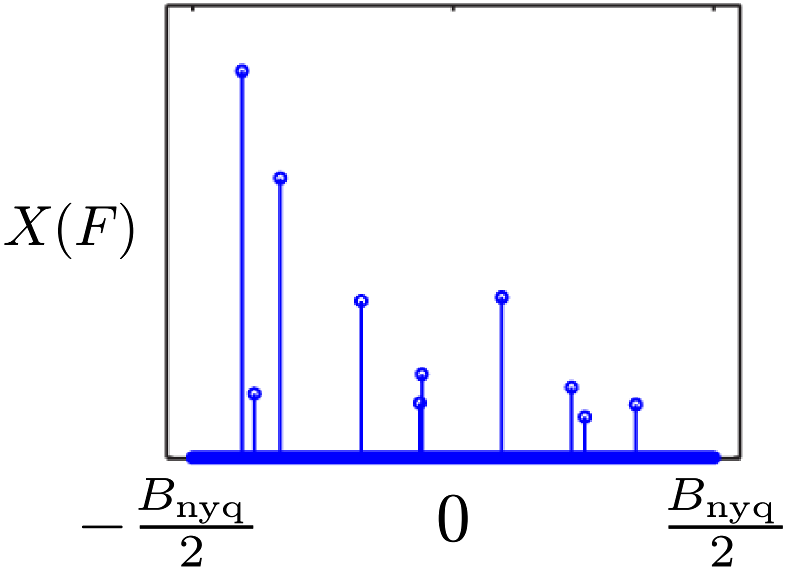
Challenge 2

Map analog sparsity into a sparsifying dictionary



Candidate Analog Signal Models

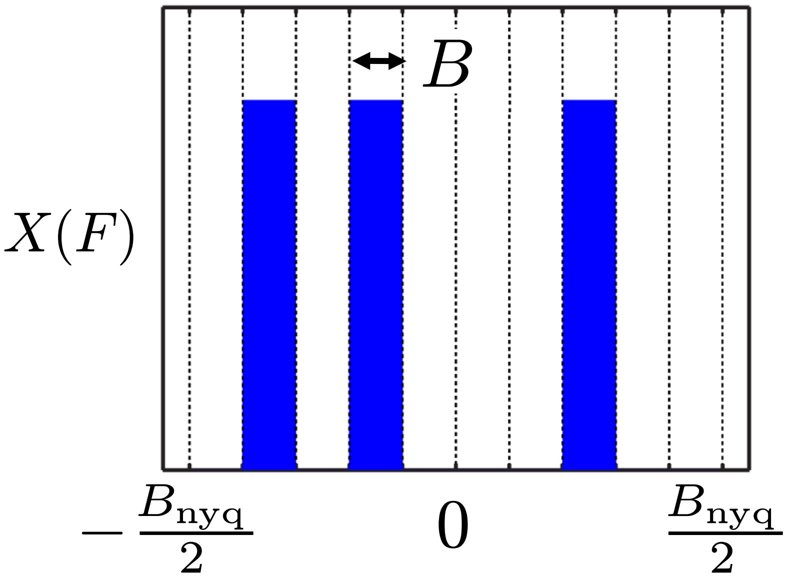
	Model for $x(t)$	Sparsifying dictionary for x	Sparsity level for x
multitone	sum of S tones	overcomplete DFT?	S-sparse



- Typical model in CS
- Coherence
- “Off-grid” tones

Candidate Analog Signal Models

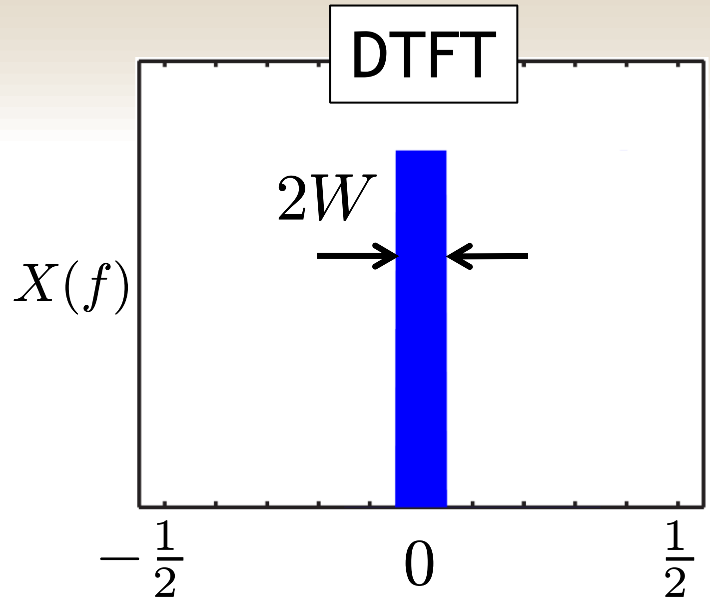
	Model for $x(t)$	Sparsifying dictionary for x	Sparsity level for x
multitone	sum of S tones	overcomplete DFT?	S-sparse
multiband	sum of K bands	?	?



- Landau
- Bresler, Feng, Venkataramani
- Eldar and Mishali

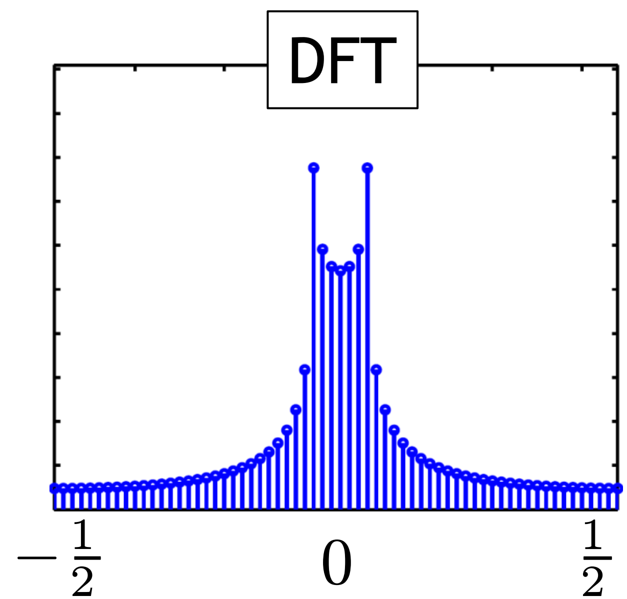
The Problem with the DFT

$$x[n] = \int_{-W}^{W} X(f) e^{j2\pi f n}\, df, \ \forall n$$



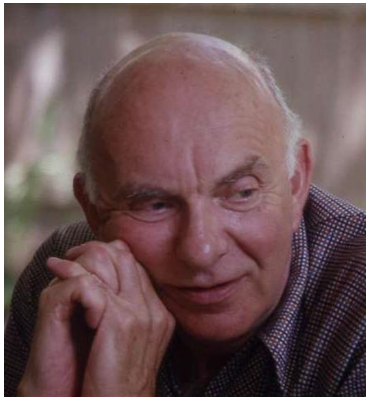
time-limiting

$$x = \sum_{k=0}^{N-1} X_k e_{\frac{k}{N}}, \quad e_f := \begin{bmatrix} e^{j2\pi f 0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f (N-1)} \end{bmatrix}$$



NOT SPARSE

Discrete Prolate Spheroidal Sequences (DPSS's)



Slepian [1978]: Given an integer N and $W \leq \frac{1}{2}$, the *DPSS's* are a collection of N vectors

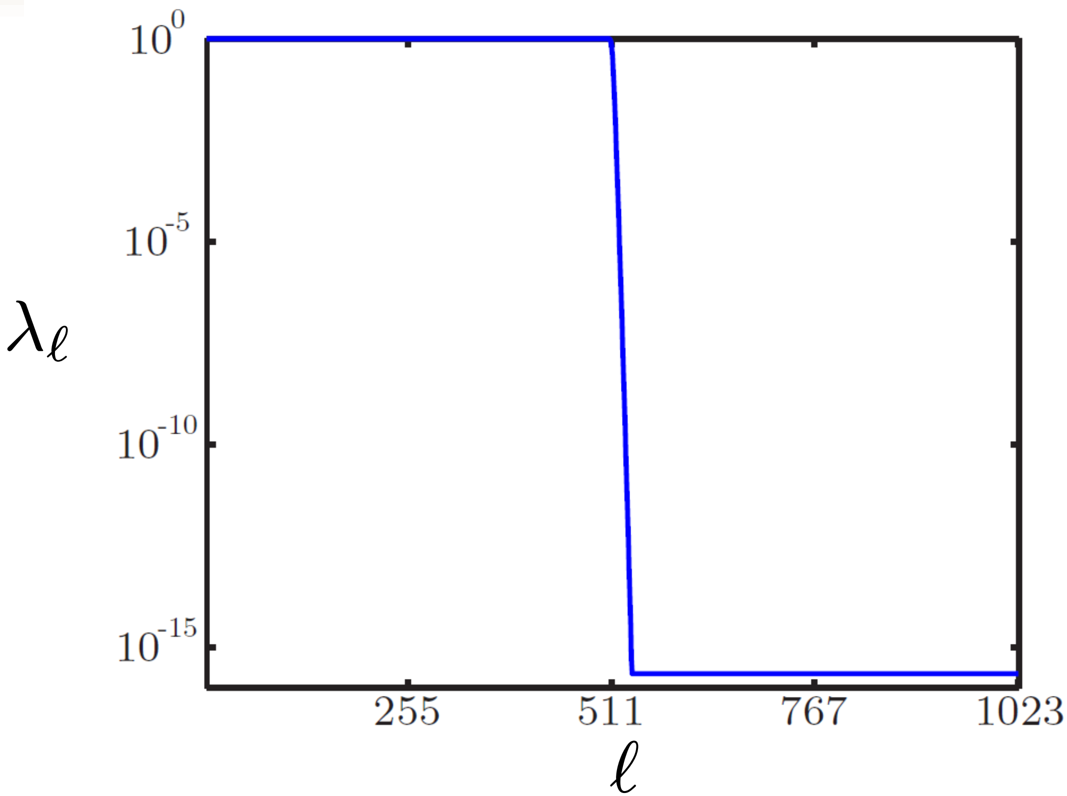
$$s_0, s_1, \ldots, s_{N-1} \in \mathbb{R}^N$$

that satisfy

$$\mathcal{T}_N(\mathcal{B}_W(s_\ell))) = \lambda_\ell s_\ell.$$

The DPSS's are perfectly time-limited, but when $\lambda_\ell \approx 1$ they are highly concentrated in frequency.

DPSS Eigenvalue Concentration



$N = 1024$

$W = \frac{1}{4}$

$2NW = 512$

The first $\approx 2NW$ eigenvalues ≈ 1.
The remaining eigenvalues ≈ 0.

Another Perspective: Subspace Fitting

$$e_f := \begin{bmatrix} e^{j2\pi f0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f(N-1)} \end{bmatrix}$$

Suppose that we wish to minimize

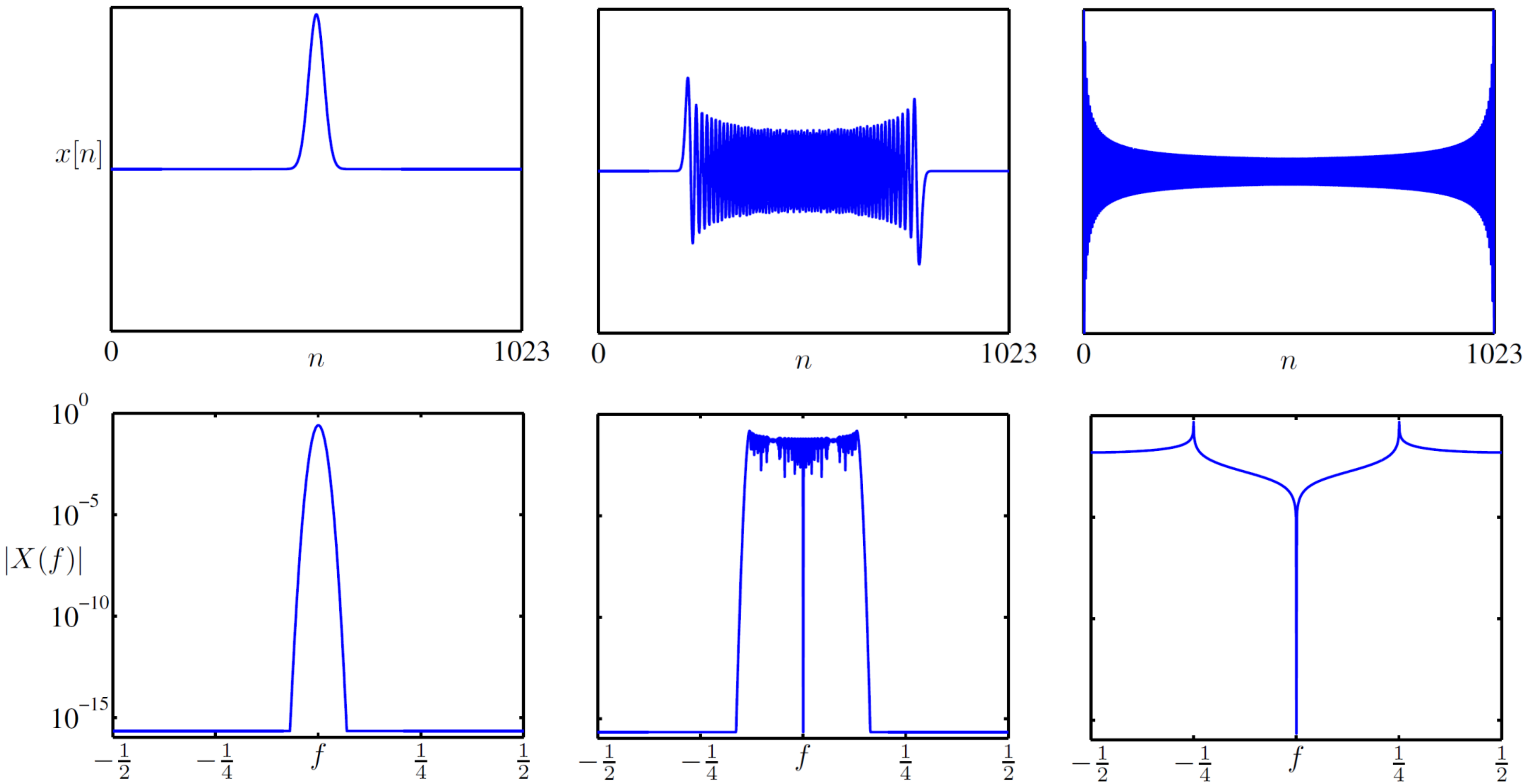
$$\int_{-W}^{W} \|e_f - P_Q e_f\|_2^2 \, df$$

over all subspaces Q of dimension k.

Optimal subspace is spanned by the first k "DPSS vectors".

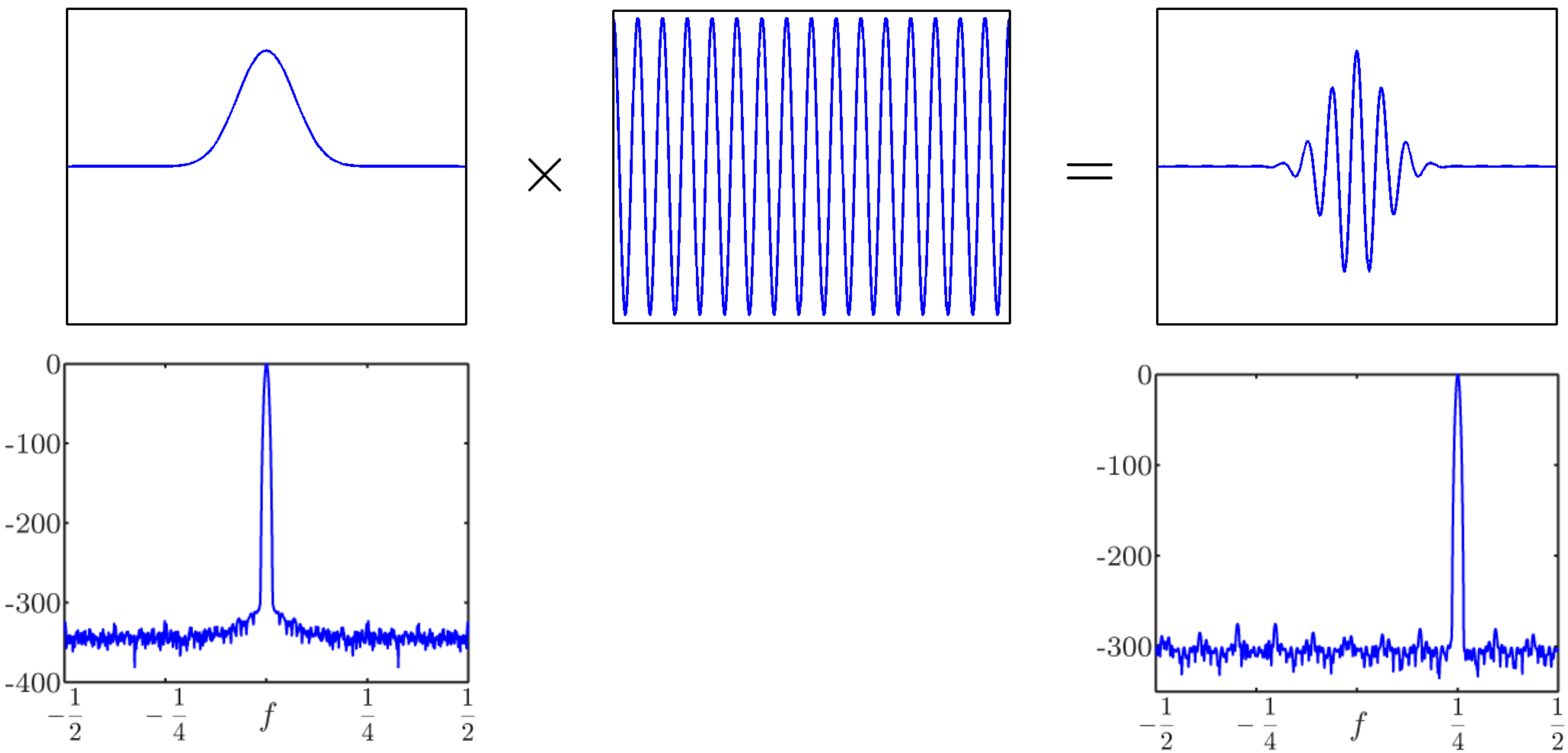
DPSS Examples

$$N = 1024 \qquad W = \frac{1}{4}$$



$\ell = 0$ $\quad$ $\ell = 127$ $\quad$ $\ell = 511$

DPSS's for Bandpass Signals

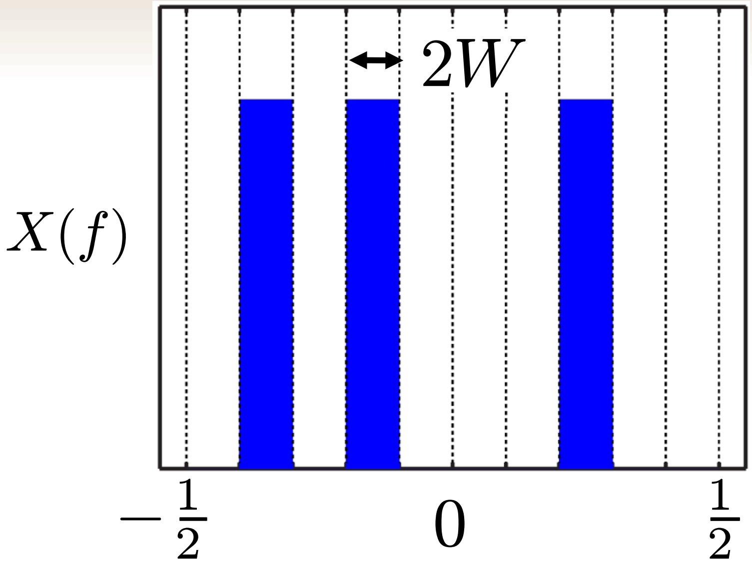


DPSS Dictionaries for CS

Modulate k DPSS vectors to center of each band:

$$D = [D_1, D_2, \ldots, D_J]$$

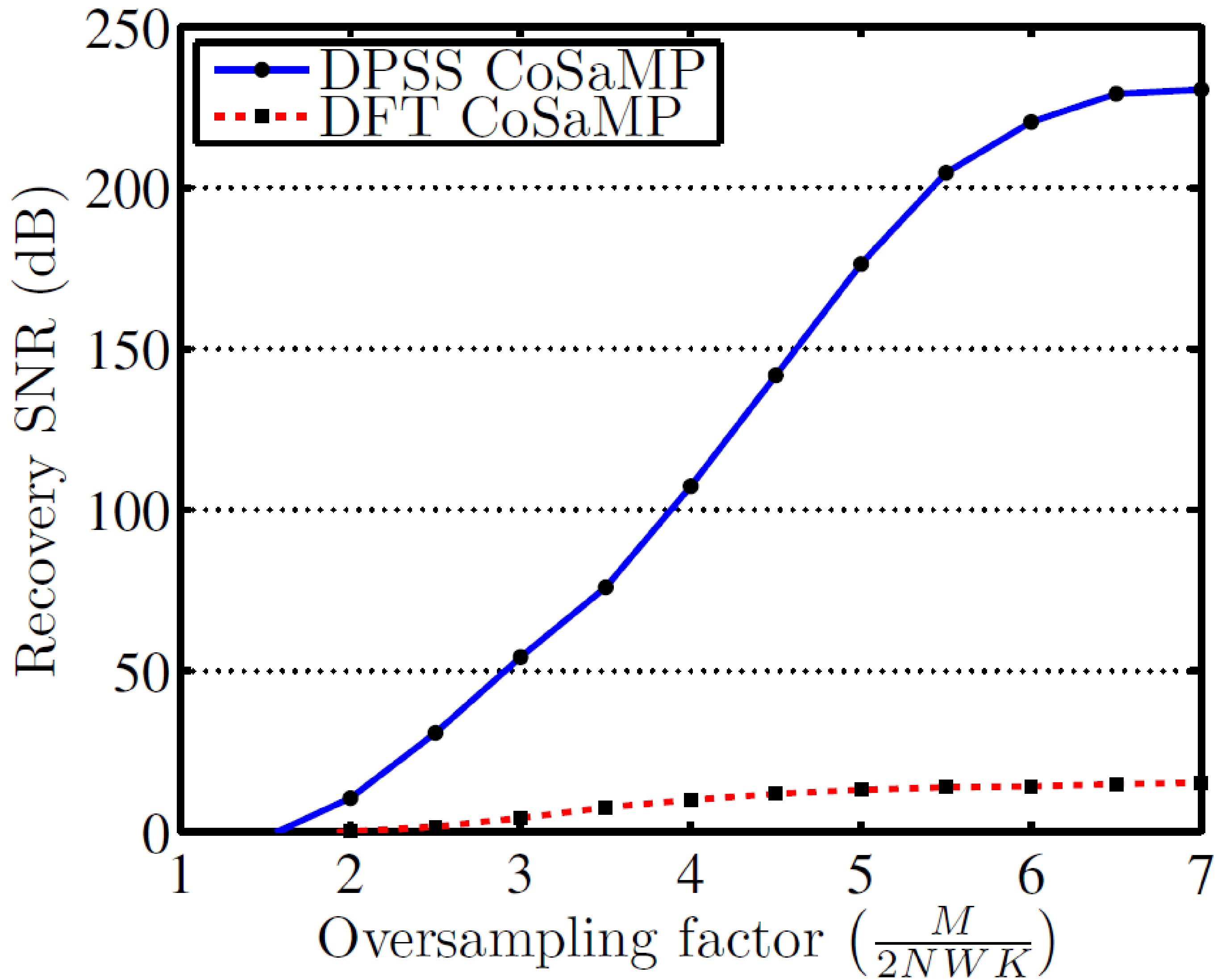
approximately square if $k \approx 2NW$



J possible bands

Most multiband signals, when sampled and time-limited, are well-approximated by a sparse representation in D.

Empirical Results: DFT Comparison



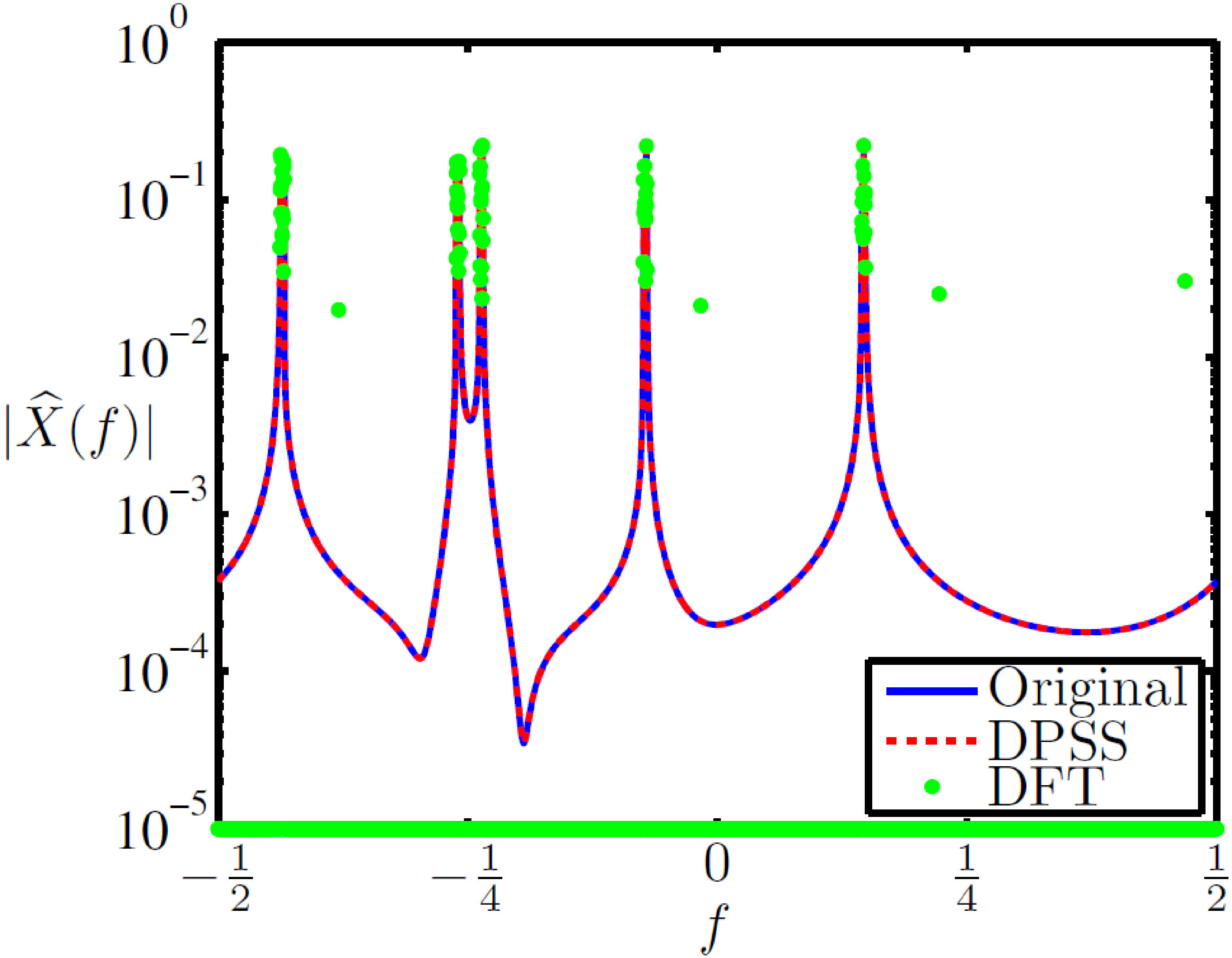
$N = 4096$

$\frac{B}{B_{\text{nyq}}} = \frac{1}{256}$

$K = 5$

[Davenport and Wakin - 2012]

Empirical Results: DFT Comparison



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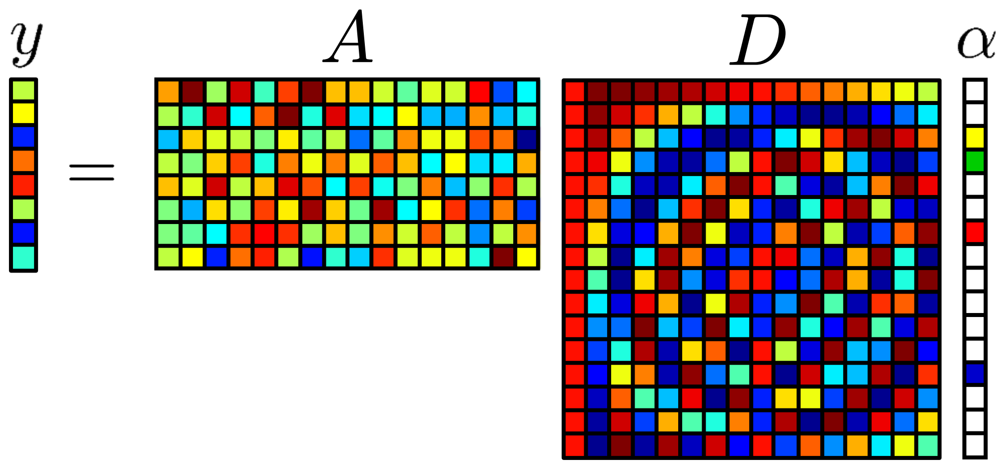
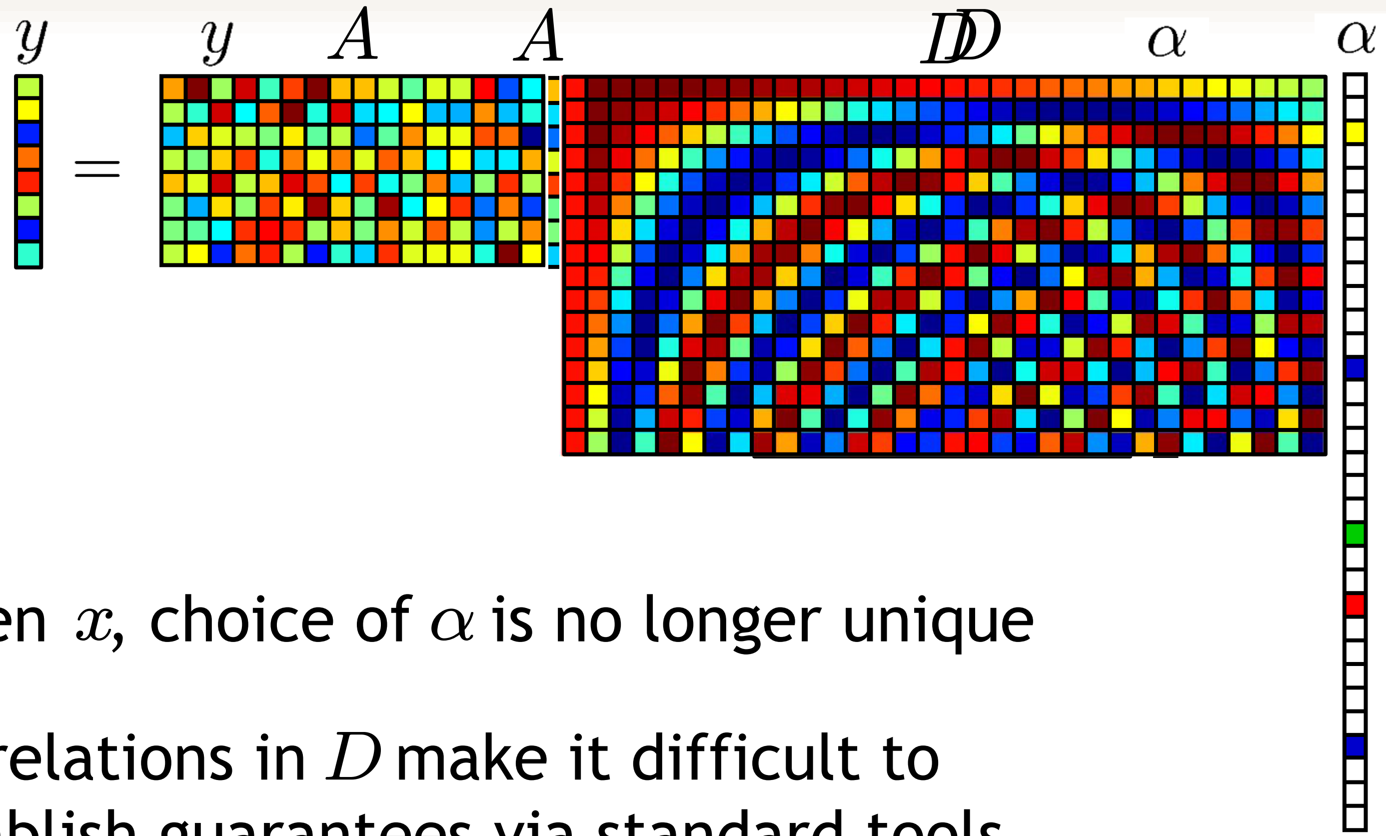
Recovery Guarantees?



$$y \rightarrow \widehat{\alpha} \rightarrow \widehat{x} = D\widehat{\alpha}$$

The Treachery of Images

The Treachery of α



- Given x, choice of α is no longer unique
- Correlations in D make it difficult to establish guarantees via standard tools
- If D is poorly conditioned, we can have $\|D\widehat{\alpha} - D\alpha\|_2 \gg \|\widehat{\alpha} - \alpha\|_2$ or $\|D\widehat{\alpha} - D\alpha\|_2 \ll \|\widehat{\alpha} - \alpha\|_2$

Signal-focused Recovery Strategy

- Focus on x instead of α

- Measure error in terms of $\|\widehat{x} - x\|_2$ instead of $\|\widehat{\alpha} - \alpha\|_2$

$$\sqrt{1-\delta_k}\|\alpha\|_2 \cdot \quad \|AD\alpha\|_2 \cdot \quad \sqrt{1+\delta_k}\|\alpha\|_2$$

$$\sqrt{1-\delta_k}\|D\alpha\|_2 \cdot \quad \|AD\alpha\|_2 \cdot \quad \sqrt{1+\delta_k}\|D\alpha\|_2$$

CoSaMP

initialize: $r = y, x^0 = 0, \ell = 0, \Gamma = \emptyset$

until converged:

proxy: $h = A^* r$

identify: $\quad = \{2S \text{ largest elements of } |h|\}$

merge: $T = \quad \cup \Gamma$

update:
$$\widetilde{x} = \arg\min_{\text{supp}(z) \subseteq T} \|y - Az\|_2$$
$$\Gamma = \{S \text{ largest elements of } |\widetilde{x}|\}$$
$$x^{\ell+1} = \widetilde{x}|_\Gamma$$
$$r^{\ell+1} = y - Ax^{\ell+1}$$
$$\ell = \ell + 1$$

output: $\widehat{x} = x^\ell$

Key Steps

$$= \{2S \text{ largest elements of } |h|\}$$

$$\widetilde{x} = \underset{\text{supp}(z) \subseteq T}{\arg\min} \|y - Az\|_2$$

$$\Gamma = \{S \text{ largest elements of } |\widetilde{x}|\}$$

$$x^{\ell+1} = \widetilde{x}|_\Gamma$$

Given a vector in $\mathbb{R}^n$, use hard thresholding to find best sparse approximation

$\mathcal{P}_\Lambda$: orthogonal projector onto $\mathcal{R}(D_\Lambda)$

$$\Lambda_{\text{opt}}(z, S) = \underset{|\Lambda|=S}{\arg\min} \|z - \mathcal{P}_\Lambda z\|_2$$

Approximate Projection

$\mathcal{P}_\Lambda$: orthogonal projector onto $\mathcal{R}(D_\Lambda)$

$$\Lambda_{\text{opt}}(z, S) = \arg\min_{|\Lambda|=S} \|z - \mathcal{P}_\Lambda z\|_2$$

$\mathcal{S}(z, S)$: estimate of $\Lambda_{\text{opt}}(z, S)$

$$\underbrace{\|\mathcal{P}_{\Lambda_{\text{opt}}} z - \mathcal{P}_{\mathcal{S}} z\|_2}_{} \cdot \quad \min\left(\epsilon_1 \|\mathcal{P}_{\Lambda_{\text{opt}}} z\|_2, \ \epsilon_2 \|z - \mathcal{P}_{\Lambda_{\text{opt}}} z\|_2\right)$$

measure quality of approximation in
“signal space”, not “coefficient space”

Signal Space CoSaMP

initialize: $r = y, x^0 = 0, \ell = 0, \Gamma = \emptyset$

until converged:

proxy: $h = A^* r$

identify: $\quad = \mathcal{S}(h, 2S)$

merge: $T = \quad \cup \, \Gamma$

update: $\widetilde{x} = \underset{z \in \mathcal{R}(D_T)}{\arg\min} \|y - Az\|_2$

$\Gamma = \mathcal{S}(\widetilde{x}, S)$

$x^{\ell+1} = \mathcal{P}_\Gamma(\widetilde{x})$

$r^{\ell+1} = y - Ax^{\ell+1}$

$\ell = \ell + 1$

output: $\widehat{x} = x^\ell$

Recovery Guarantees

Suppose there exists an S-sparse α such that $x = D\alpha$ and A satisfies the D-RIP of order $4S$.

If we observe $y = Ax + e$, then

$$\|x - x^{\ell+1}\|_2 \cdot \; C_1\|x - x^{\ell}\|_2 + C_2\|e\|_2$$

For $\delta_{4k} = 0.029, \epsilon_1 = 0.1, \epsilon_2 = 1,$

$$\|x - x^{\ell}\|_2 \cdot \; 2^{-\ell}\|x\|_2 + 25.4\|e\|_2$$

[Davenport, Needell, and Wakin - 2013]

Practical Choices for $\mathcal{S}(z, S)$

Given z, we want to find an S-sparse α such that $z \approx D\alpha$

- Any sparse recovery algorithm!
- CoSaMP
- Orthogonal Matching Pursuit (OMP)
- ℓ_1-minimization followed by hard-thresholding

$$\mathcal{S}(z, S) = H_S\left(\arg\min_{w: Dw = z} \|w\|_1\right)$$

Remaining Gaps

- None of the "practical choices" are proven to provide the desired approximate projections

- Experimental results suggest that (at least for certain dictionaries) none of these choices are sufficient

- Recent progress
  - Hegde and Indyk (2013)
  - Giryes and Needell (2013)
  - weaker requirements on approximate projection

Conclusion

- Dealing with analog signals in the traditional compressive sensing framework requires
  - new sparsifying dictionaries
  - modified algorithms
  - *signal-focused* analysis

- Many open questions remain
  - provably near-optimal algorithms for computing approximate projections
  - may actually involve the use of DPSS's
  - efficient methods for handling both multiband and multitone signals simultaneously

Thank You!