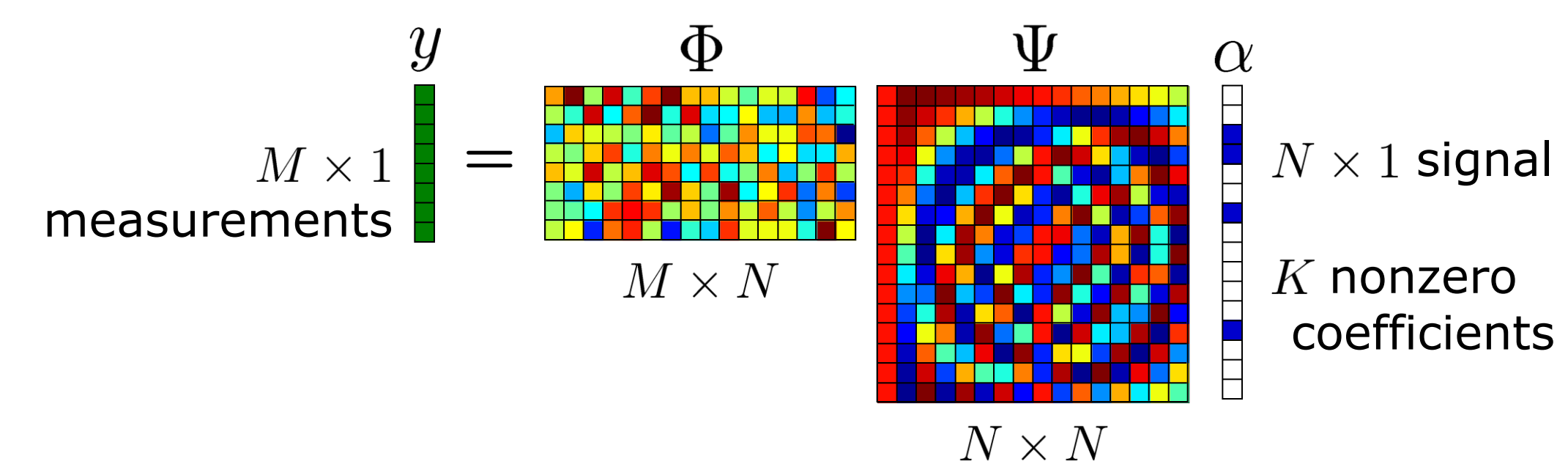


Compressive Sensing

Background

Directly acquire a reduced set of low-dimensional **compressive measurements**



Nonlinear recovery via optimization-based, iterative, or greedy algorithms

Basis Pursuit (BP)

$$\hat{\alpha} = \arg \min_{\alpha} \|\alpha\|_1 \quad \text{s.t. } y = \Phi\Psi\alpha$$

Basis Pursuit De-Noising (BPDN)

$$\hat{\alpha} = \arg \min_{\alpha} \|\alpha\|_1 \quad \text{s.t. } \|y - \Phi\Psi\alpha\|_2 \leq \epsilon$$

The **restricted isometry property** (RIP) ensures that Φ captures the information in the signal

$$(1 - \delta)\|\alpha\|_2^2 \leq \|\Phi\Psi\alpha\|_2^2 \leq (1 + \delta)\|\alpha\|_2^2 \quad \forall \alpha \quad \|\alpha\|_0 \leq K$$

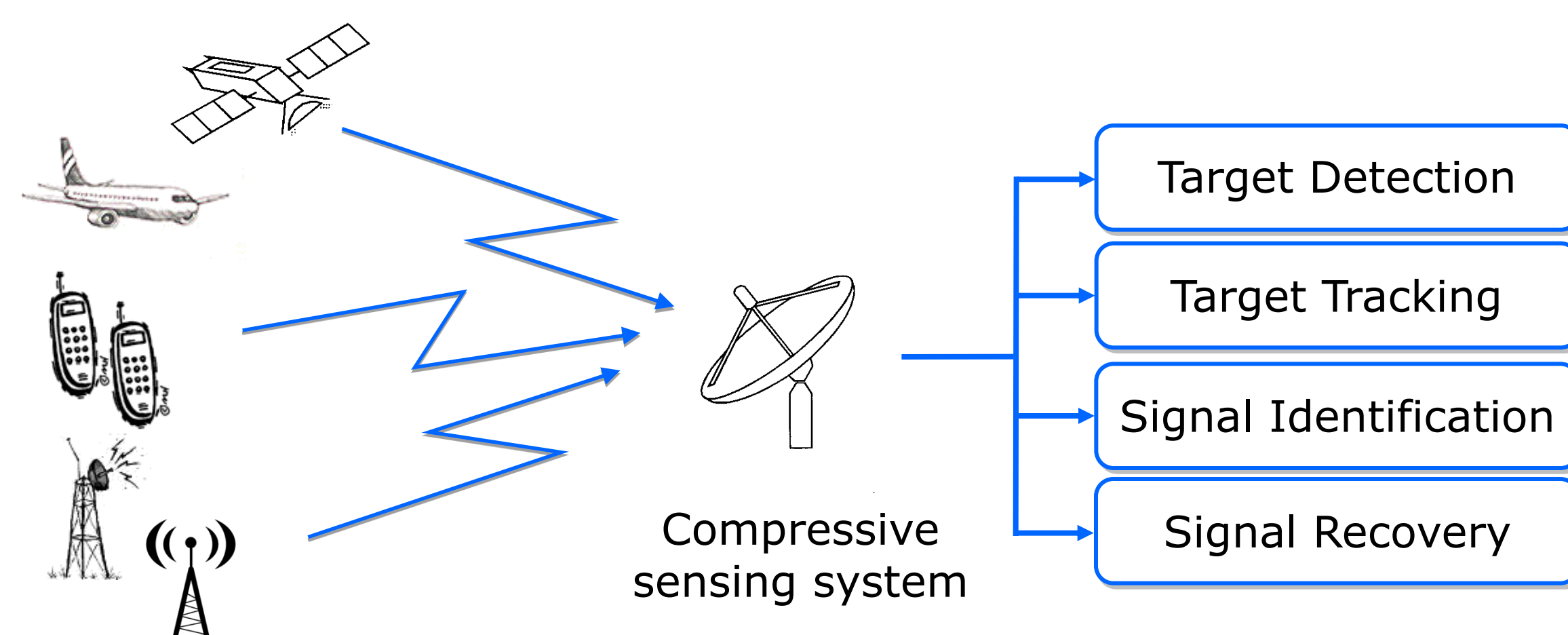
Theorem [Candès]: Suppose that $y = \Phi\Psi\alpha + e$ and $\|\alpha\|_0 \leq K$. If Φ satisfies the RIP of order $2K$ with $\delta < \sqrt{2} - 1$, then the BPDN solution satisfies $\|\hat{\alpha} - \alpha\|_2 \leq C_0\|e\|_2$.

SubGaussian Φ satisfy the RIP if $M = O(K \log(N/K))$.

Does randomness provide any other benefits?

Compressive Signal Processing

Random measurements are **information scalable**



In such scenarios, measurements are often corrupted by **interference** and **structured noise**

$$y = \Phi x_S + \Phi x_I \quad y = \Phi x_S + \Omega e$$

Seek to remove contribution of Φx_I or Ωe to y before reconstructing x_S .

Corruption

Interference Cancellation

Assume $x_S \in \mathcal{X}_S$ and $x_I \in \mathcal{X}_I$, where $\langle x_I, x_S \rangle = 0$ for all $x_S \in \mathcal{X}_S, x_I \in \mathcal{X}_I$. Also assume $\Psi = I$.

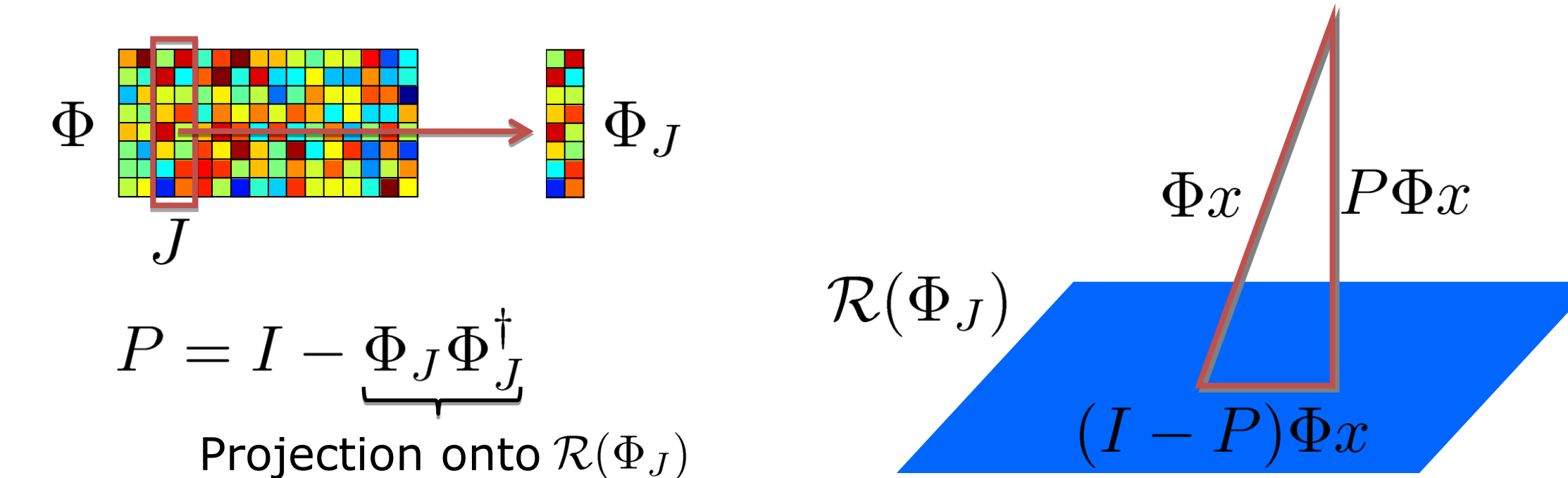
Design $M \times M$ matrix P such that

$$\|P(\Phi x_I)\|_2 \approx 0 \quad \text{and} \quad \|P(\Phi x_S)\|_2 \approx \|\Phi x_S\|_2$$

Note: Not always possible
Depends on structure of \mathcal{X}_S and \mathcal{X}_I

Subspace Cancellation

Example: x_I has known support set J of size K_I
Seek P such that $\mathcal{R}(\Phi_J) \subset \mathcal{N}(P)$.



Observe that $P y = P \Phi x_S + P \Phi x_I = P \Phi x_S$

Theorem: If Φ satisfies the RIP of order $2K_S + K_I$ then $P\Phi$ satisfies

$$\left(1 - \frac{\delta}{1 - \delta}\right) \|x\|_2^2 \leq \|P\Phi x\|_2^2 \leq (1 + \delta) \|x\|_2^2$$

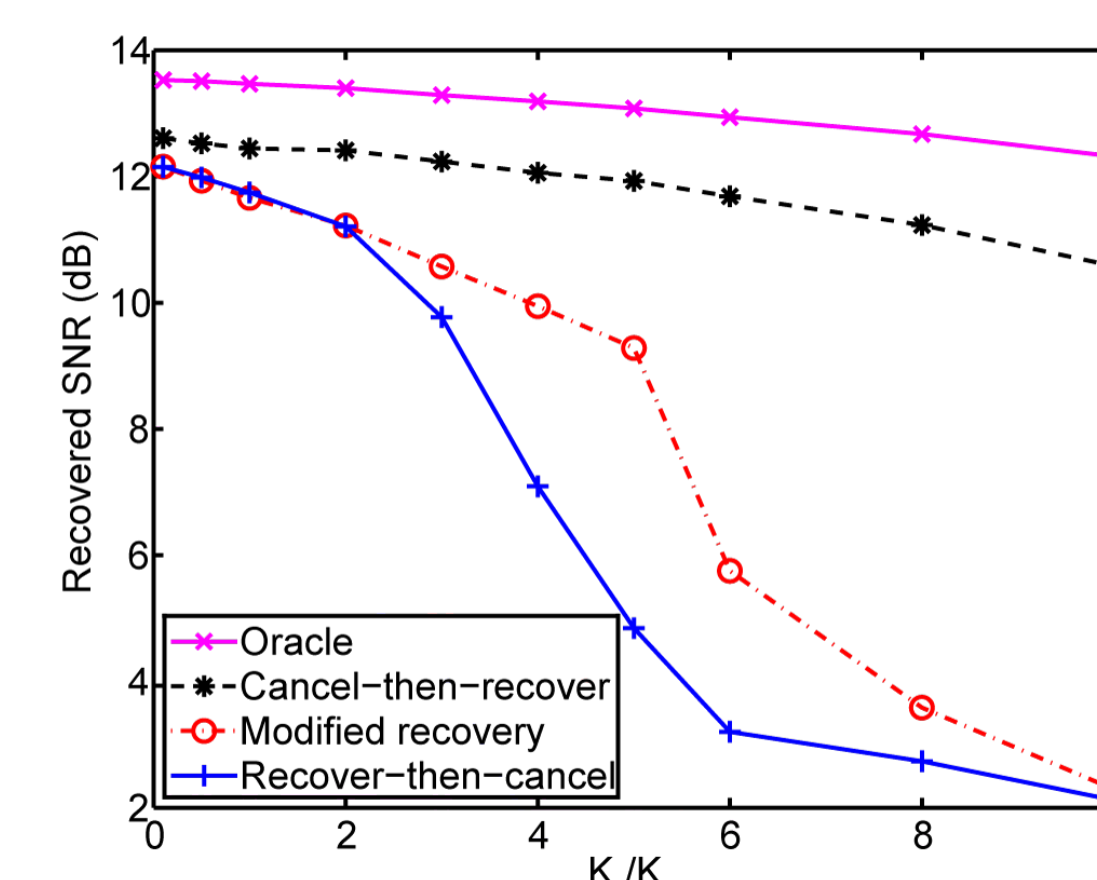
for all x such that $\|x\|_0 \leq 2K_S$ and $\text{supp}(x) \cap J = \emptyset$.

Proof exploits two facts:

$$\begin{aligned} \|\Phi x\|_2^2 &= \|P\Phi x\|_2^2 + \|(I - P)\Phi x\|_2^2 \\ \frac{\|(I - P)\Phi x\|_2}{\|\Phi x\|_2} &= \frac{\langle (I - P)\Phi x, \Phi x \rangle}{\|(I - P)\Phi x\|_2 \|\Phi x\|_2} \leq \frac{\delta}{1 + \delta} \end{aligned}$$

Implications

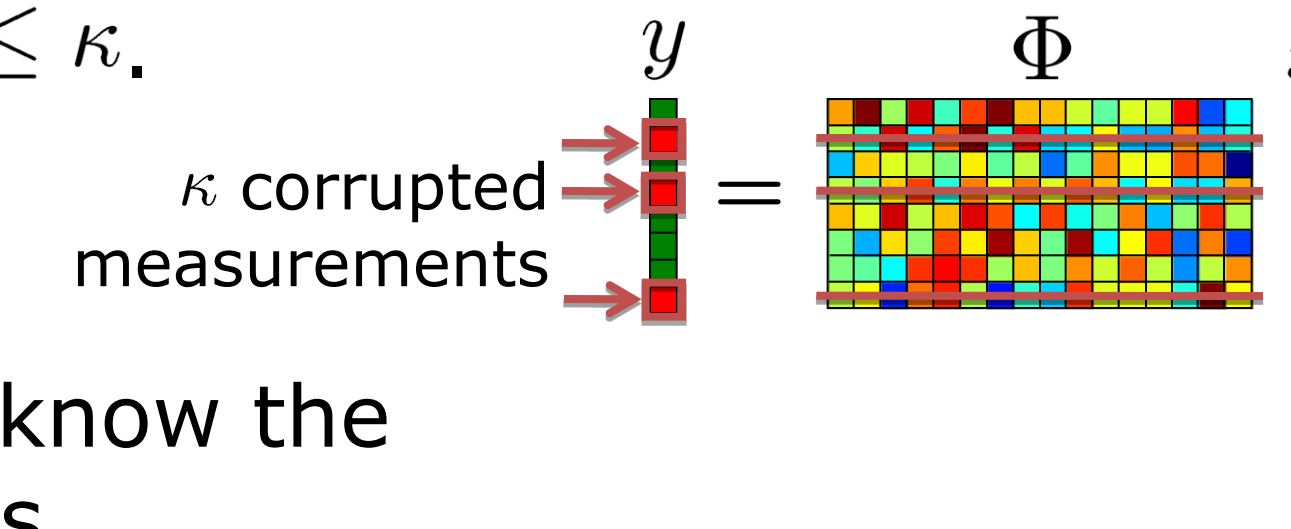
- Allows for **"cancel-then-recover"** approach to signal recovery in interference
- Useful tool in analysis of greedy algorithms such as **Orthogonal Matching Pursuit** (OMP) and **Regularized OMP** (ROMP)



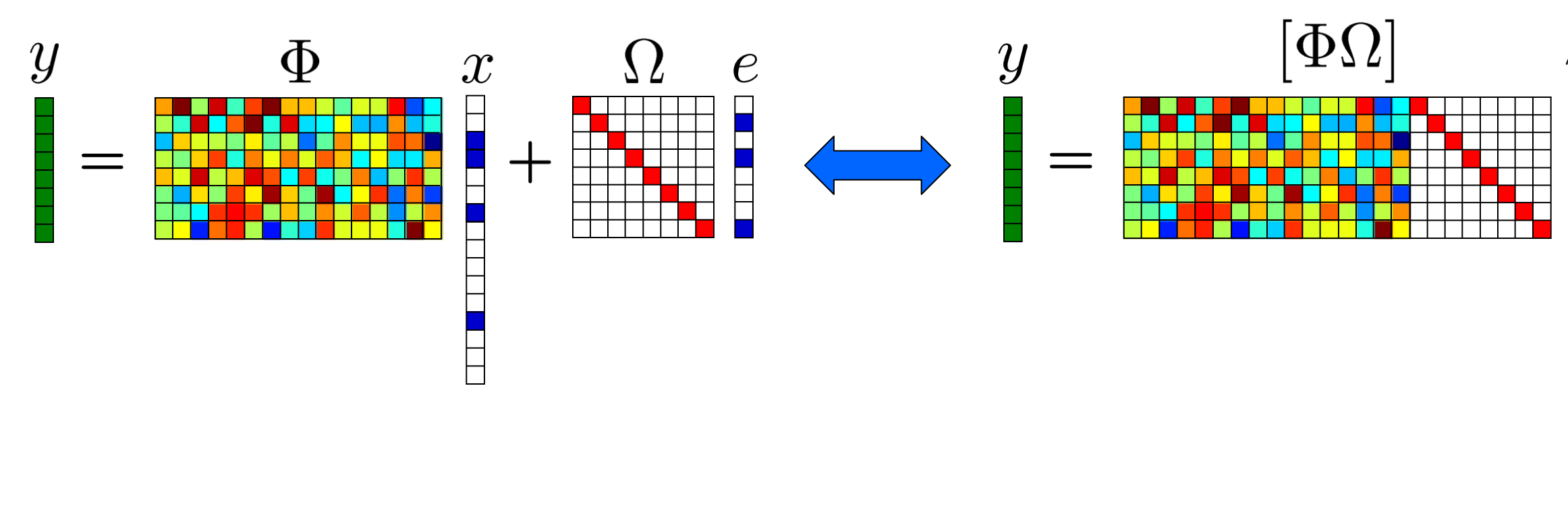
Justice

Corrupted Measurements

Suppose $y = \Phi x + \Omega e$ where $\Omega = I$ and $\|e\|_0 \leq \kappa$.



In general, we do not know the locations of corruptions.



"Justice Pursuit" (JP)

$$\hat{u} = \arg \min_u \|u\|_1 \quad \text{s.t. } y = [\Phi \Omega] u$$

To analyze Justice Pursuit, we must study the properties of the matrix $[\Phi \Omega]$.

Theorem: If Φ is a subGaussian matrix with

$$M = O\left((K + \kappa) \log\left(\frac{N + M}{K + \kappa}\right)\right)$$

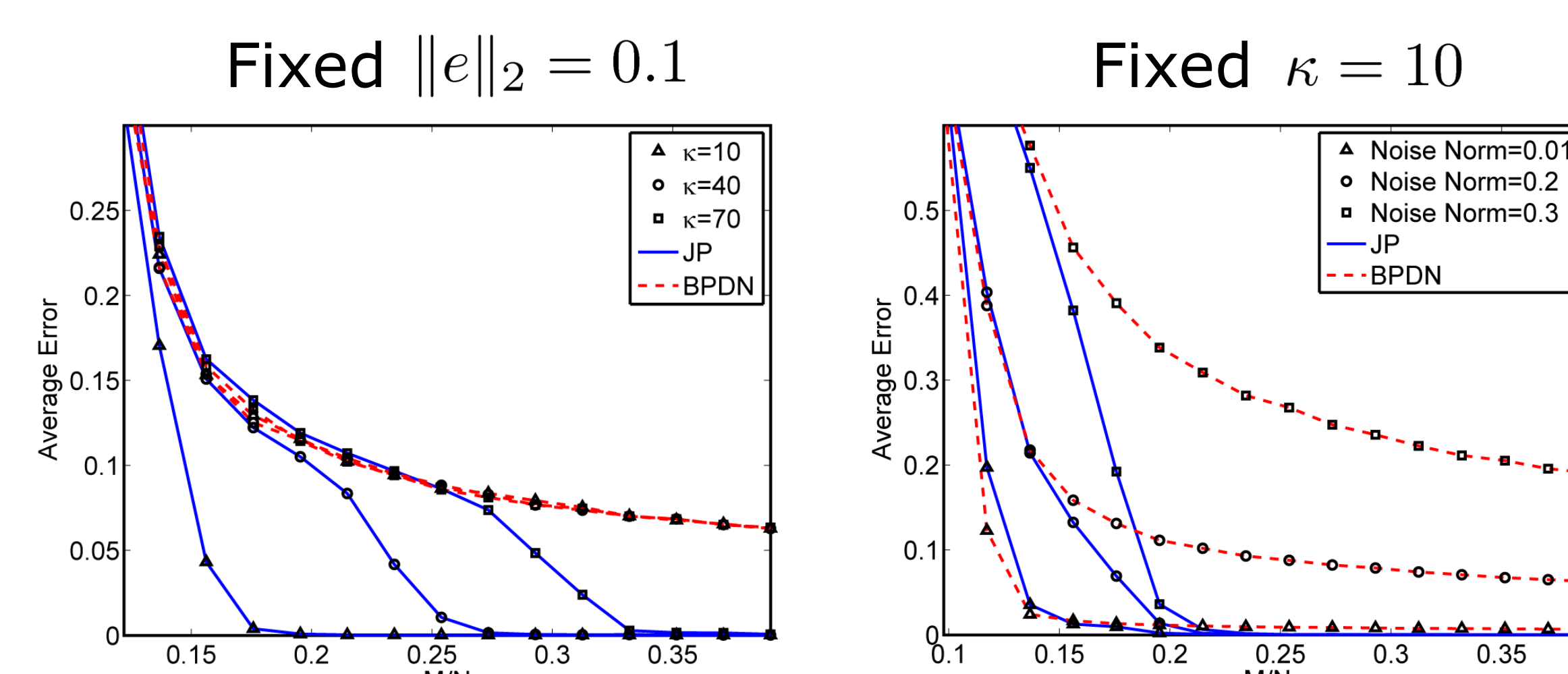
then $[\Phi \Omega]$ satisfies the RIP of order $(K + \kappa)$ with probability at least $1 - 3e^{-CM}$.

Proof follows from

$$\begin{aligned} \|[\Phi \Omega] u\|_2^2 &= \|\Phi x\|_2^2 + e^T \Omega^T \Phi x + \|e\|_2^2 \\ \text{and the facts that with high probability} \\ -\delta \|e\|_2 \|x\|_2 &\leq e^T \Omega^T \Phi x \leq \delta \|e\|_2 \|x\|_2 \\ (1 - \delta) \|x\|_2^2 &\leq \|\Phi x\|_2^2 \leq (1 + \delta) \|x\|_2^2 \end{aligned}$$

Experiments

Compare JP with BPDN ($N = 1024, K = 10$)
If M is sufficiently large, JP achieves **exact** recovery



Democracy

Corruption meets Justice

The key results of subspace cancellation and justice combine to provide a simple proof that random matrices are **democratic**.

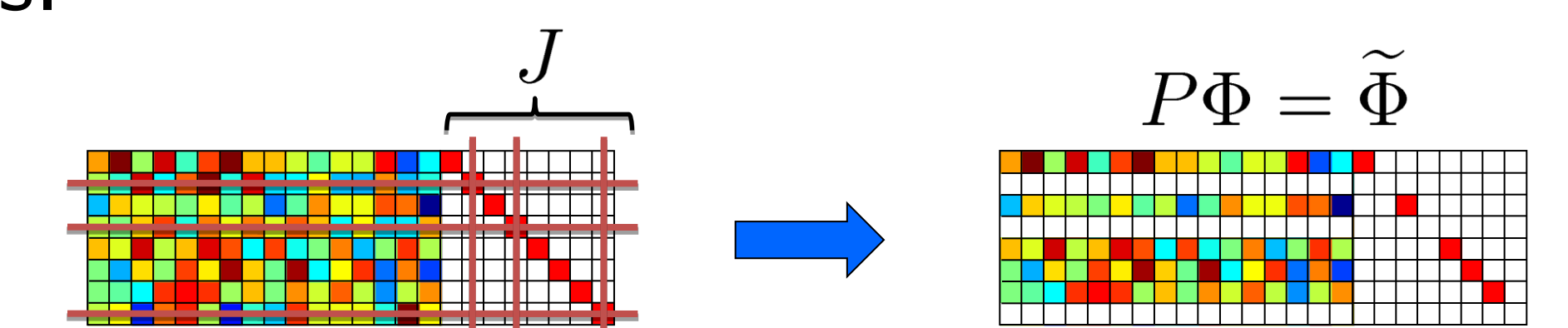
A matrix is democratic if we can remove D arbitrary (adversarially selected) rows and retain the RIP.

If

$$M = O\left((K + D) \log\left(\frac{N + M}{K + D}\right)\right)$$

then $[\Phi I]$ satisfies the RIP of order $(K + D)$.

Construct P to cancel interference from columns indexed by J , where J corresponds to a set of D rows.



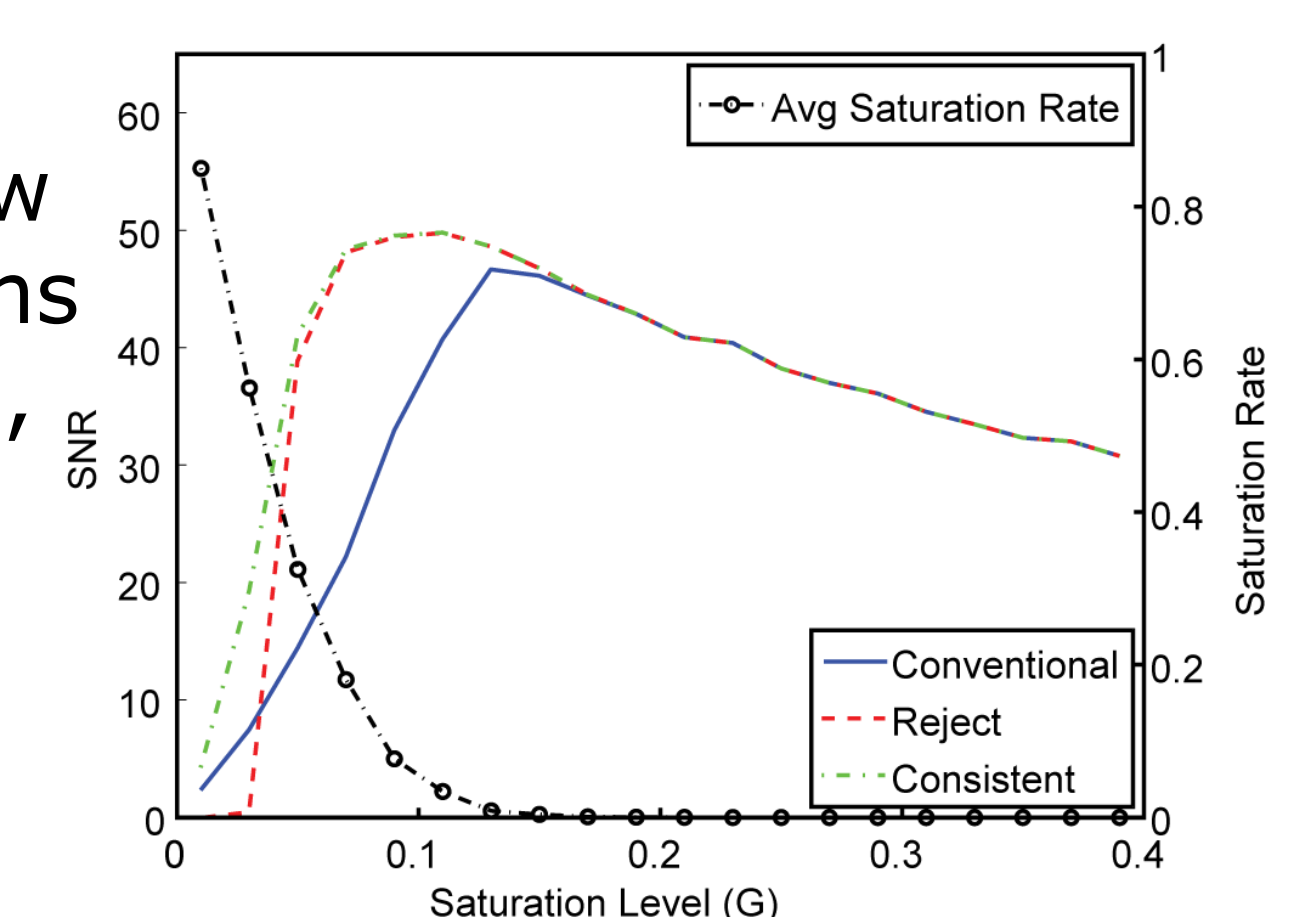
Since $\tilde{\Phi}$ will satisfy the RIP for any possible choice of J , this establishes that Φ is democratic.

Democracy in Action

When measurements are quantized using a finite-range quantizer, some will **saturate**.

Democracy justifies a strategy of simply **rejecting** saturated measurements.

In fact, simulations show this method out-performs the traditional approach, achieving optimal performance at non-zero saturation rates.



References

- M.A. Davenport, P.T. Boufounos, M.B. Wakin, and R.G. Baraniuk, "Signal processing with compressive measurements," to appear in *IEEE J. Selected Topics in Signal Processing*, 2010.
- M.A. Davenport and M.B. Wakin, "Analysis of Orthogonal Matching Pursuit using the restricted isometry property," Preprint, 2009.
- J.N. Laska, M.A. Davenport, and R.G. Baraniuk, "Exact signal recovery from sparsely corrupted measurements through the pursuit of justice," *Asilomar Conf. Signals, Systems, and Computers*, 2009.
- J.N. Laska, P.T. Boufounos, M.A. Davenport, and R.G. Baraniuk, "Democracy in action: Quantization, saturation, and compressive sensing," Preprint, 2009.
- M.A. Davenport, J.N. Laska, P.T. Boufounos, and R.G. Baraniuk, "A simple proof that random matrices are democratic," Rice ECE Tech. Report TREE 0906, 2009.