A Simple Framework for Analog Compressive Sensing

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Compressive Sensing (CS)

\[ y = \Phi x \]

Can we really acquire analog signals with "CS"?
Potential Obstacles

**Obstacle 1:** CS is discrete, finite-dimensional

**Obstacle 2:** Analog sparse representations
If $x(t)$ is bandlimited, 

$$y[m] = \langle \phi_m(t), x(t) \rangle = \sum_{n=-\infty}^{\infty} x[n] \langle \phi_m(t), \text{sinc}(t/T_s - n) \rangle$$

Map analog sensing to matrix multiplication.
Obstacle 2

Map analog sparsity into digital sparsity

\[ x \xrightarrow{\Psi} \alpha \]

$N \times 1$ vector

Nyquist-rate samples of $x(t)$
### Candidate Analog Signal Models

<table>
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<tr>
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<th>Model for $x(t)$</th>
<th>Basis for $x$</th>
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<td>sum of $S$ tones</td>
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- Typical model in CS
- Coherence
- “Off-grid” tones
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<td>multiband</td>
<td>sum of $K$ bands</td>
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![Graph of $X(F)$]

- Landau
- Bresler, Feng, Venkataramanii
- Eldar, Mishali
The Problem with the DFT

\[ x[n] = \int_{-W}^{W} X(f) e^{j2\pi fn} \, df, \quad \forall n \]

\[ x = \sum_{k=0}^{N-1} X_k e^{\frac{jk}{N}}, \quad e^f := \begin{bmatrix} e^{j2\pi f_0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f(N-1)} \end{bmatrix} \]

NOT SPARSE
Another Perspective: Subspace Fitting

Suppose that we wish to minimize

$$
\int_{-W}^{W} \left\| e_f - P_Q e_f \right\|_2^2 \, df
$$

over all subspaces $Q$ of dimension $k$.

Optimal subspace is spanned by the first $k$ “DPSS vectors”.

$$
e_f := \begin{bmatrix}
e^{j2\pi f_0} \\
e^{j2\pi f} \\
\vdots \\
e^{j2\pi f(N-1)}
\end{bmatrix}$$
**Discrete Prolate Spheroidal Sequences (DPSS’s)**

*Slepiann [1978]:* Given an integer $N$ and $W \leq \frac{1}{2}$, the DPSS’s are a collection of $N$ vectors

$$s_0, s_1, \ldots, s_{N-1} \in \mathbb{R}^N$$

that satisfy

$$\mathcal{T}_N(\mathcal{B}_W(s_\ell))) = \lambda_\ell s_\ell.$$ 

The DPSS’s are perfectly time-limited, but when $\lambda_\ell \approx 1$ they are highly concentrated in frequency.
DPSS Eigenvalue Concentration

The first \(\approx 2NW\) eigenvalues \(\approx 1\).
The remaining eigenvalues \(\approx 0\).

\[
N = 1024 \\
W = \frac{1}{4} \\
2NW = 512
\]
DPSS Examples

$N = 1024 \quad W = \frac{1}{4}$

\[ x[n] \]

\[ |X(f)| \]

\( \ell = 0 \)

\( \ell = 127 \)

\( \ell = 511 \)
DPSS’s for Bandpass Signals
DPSS Dictionaries for CS

Modulate $k$ DPSS vectors to center of each band:

$$\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_J]$$

approximately square if $k \approx 2NW$

Most multiband signals, when sampled and time-limited, are well-approximated by a sparse representation in $\Psi$. 
Block-Sparse Recovery

Nonzero coefficients of $\alpha$ should be clustered in blocks according to the occupied frequency bands

$$x = [\Psi_1, \Psi_2, \ldots, \Psi_J] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_J \end{bmatrix}$$

This can be leveraged to reduce the required number of measurements and improve performance through “model-based CS”

- Baraniuk et al. [2008], Blumensath and Davies [2009, 2011]
- Group LASSO
Empirical Results: Noise

\[ N = 4096 \]
\[ M = 512 \]
\[ K = 5 \]
\[ \frac{B}{B_{\text{nyq}}} = \frac{1}{256} \]

[Davenport and Wakin - 2012]
Empirical Results: DFT Comparison

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Conclusions

• DPSS’s can be used to efficiently represent most sampled multiband signals
  - far superior to DFT

• Two types of error: approximation + reconstruction
  - approximation: small for most signals
  - reconstruction: tends to be small
  - delicate balance in practice, seems to be a sweet spot

• This approach combines careful design of $\Psi$ with more sophisticated sparse models
  - relevant in many contexts beyond ADCs