A Simple Framework for Analog Compressive Sensing

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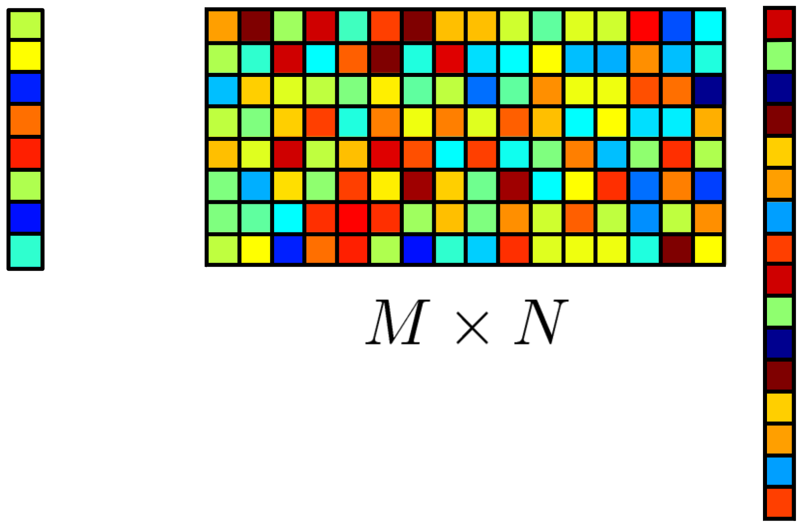
Compressive Sensing (CS)

$$y = \Phi x$$



Can we really acquire analog signals with "CS"?

Potential Obstacles

Obstacle 1: CS is discrete, finite-dimensional

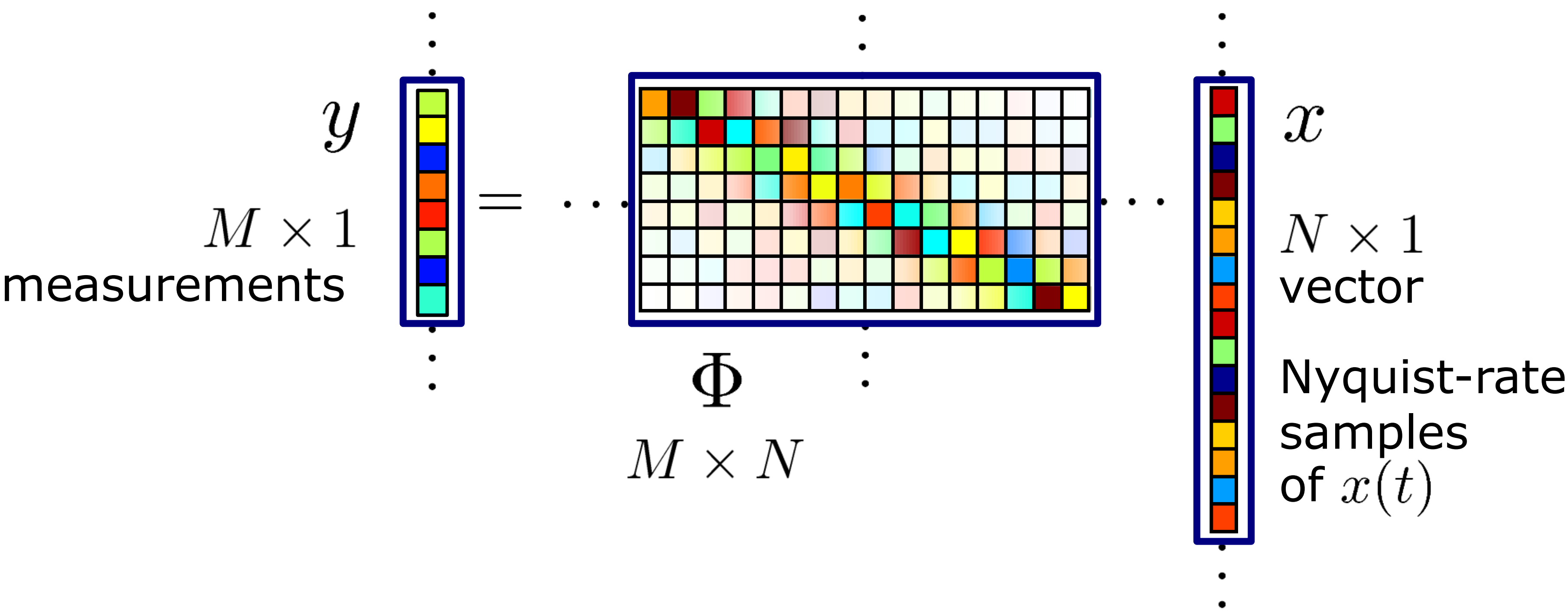
Obstacle 2: Analog sparse representations

Obstacle 1

Map analog sensing to matrix multiplication

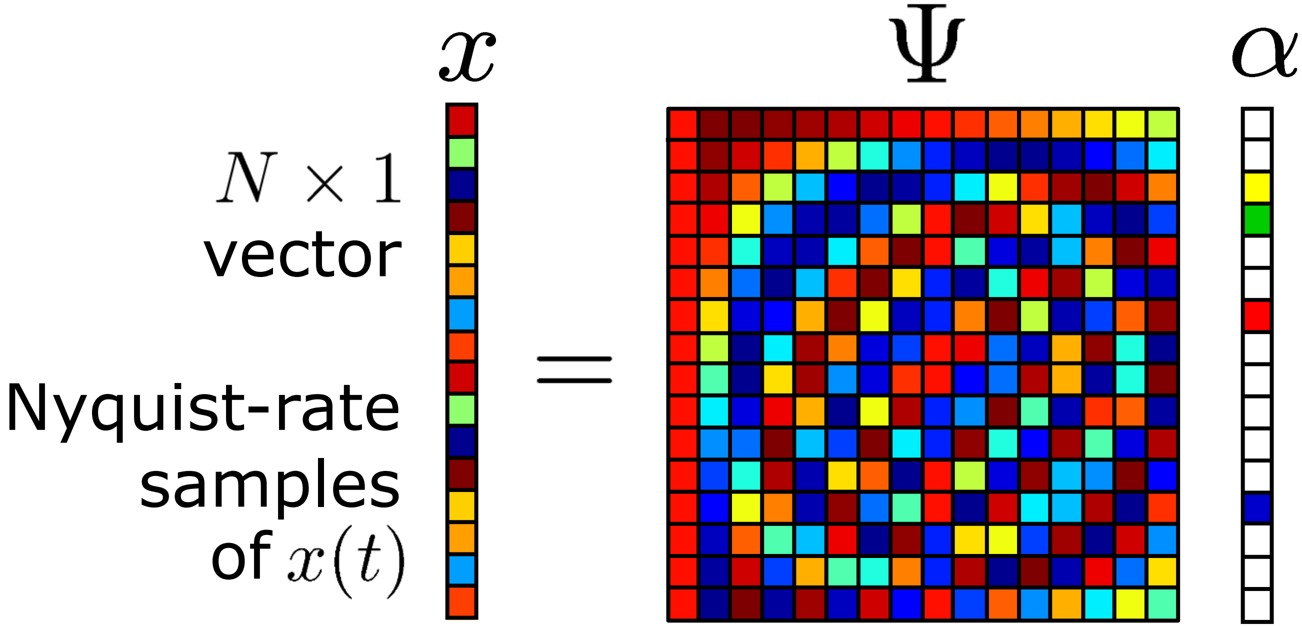
If $x(t)$ is bandlimited,

$$y[m] = \langle \phi_m(t), x(t) \rangle = \sum_{n=-\infty}^{\infty} x[n] \, \langle \phi_m(t), \mathrm{sinc}(t/T_s - n) \rangle$$



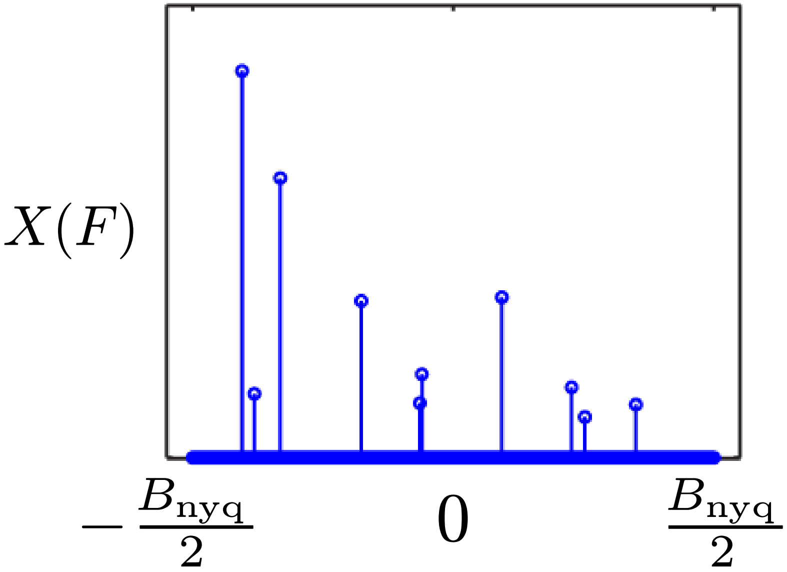
Obstacle 2

Map analog sparsity into digital sparsity



Candidate Analog Signal Models

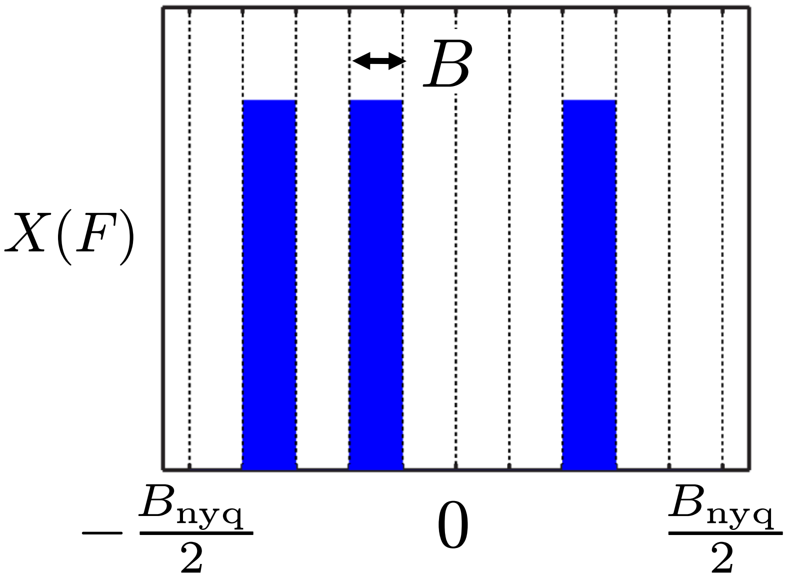
	Model for $x(t)$	Basis for x	Sparsity level for x
multitone	sum of S tones	overcomplete DFT	S -sparse



- Typical model in CS
- Coherence
- “Off-grid” tones

Candidate Analog Signal Models

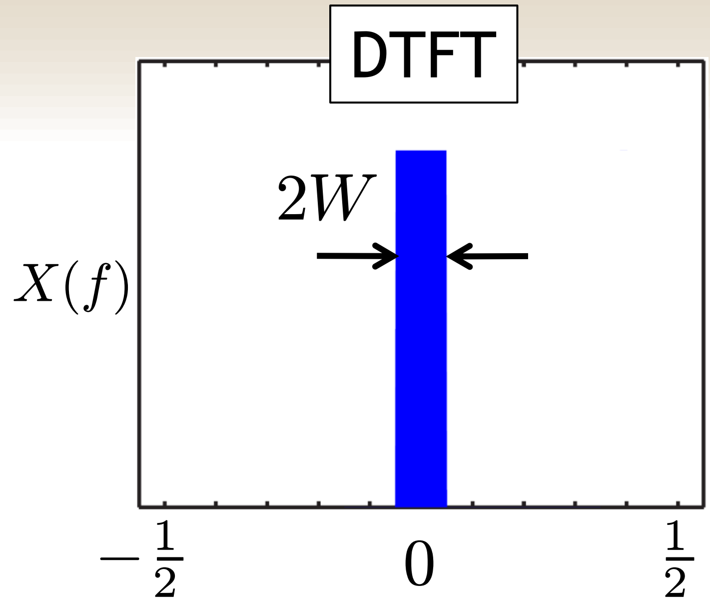
	Model for $x(t)$	Basis for x	Sparsity level for x
multitone	sum of S tones	overcomplete DFT	S -sparse
multiband	sum of K bands	?	?



- Landau
- Bresler, Feng, Venkataramani
- Eldar, Mishali

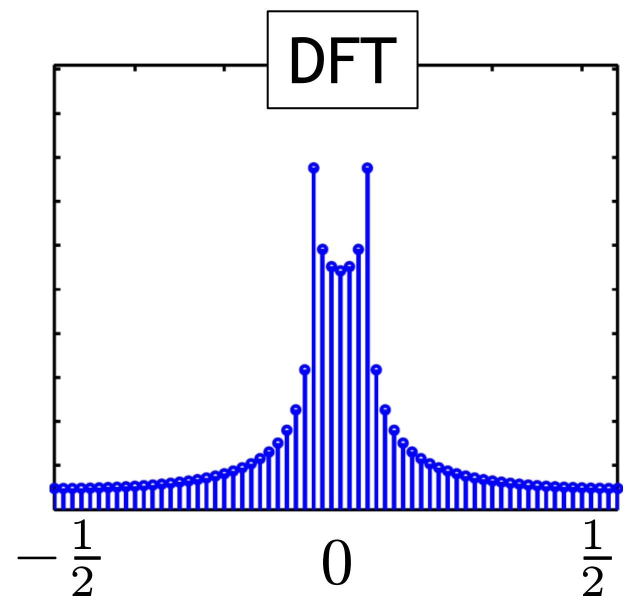
The Problem with the DFT

$$x[n] = \int_{-W}^{W} X(f) e^{j2\pi f n}\, df, \ \forall n$$



time-limiting

$$x = \sum_{k=0}^{N-1} X_k e_{\frac{k}{N}}, \quad e_f := \begin{bmatrix} e^{j2\pi f 0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f (N-1)} \end{bmatrix}$$



NOT SPARSE

Another Perspective: Subspace Fitting

$$e_f := \begin{bmatrix} e^{j2\pi f0} \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f(N-1)} \end{bmatrix}$$

Suppose that we wish to minimize

$$\int_{-W}^{W} \|e_f - P_Q e_f\|_2^2 \, df$$

over all subspaces Q of dimension k .

Optimal subspace is spanned by the first k "DPSS vectors".

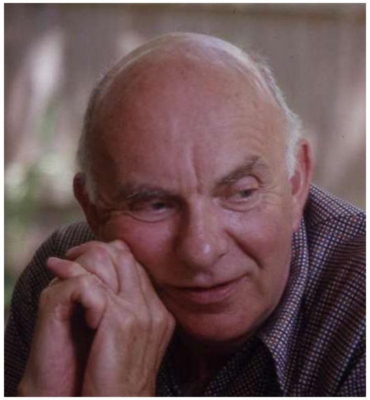
Discrete Prolate Spheroidal Sequences (DPSS's)

Slepian [1978]: Given an integer N and $W \leq \frac{1}{2}$, the *DPSS's* are a collection of N vectors

$$s_0, s_1, \ldots, s_{N-1} \in \mathbb{R}^N$$

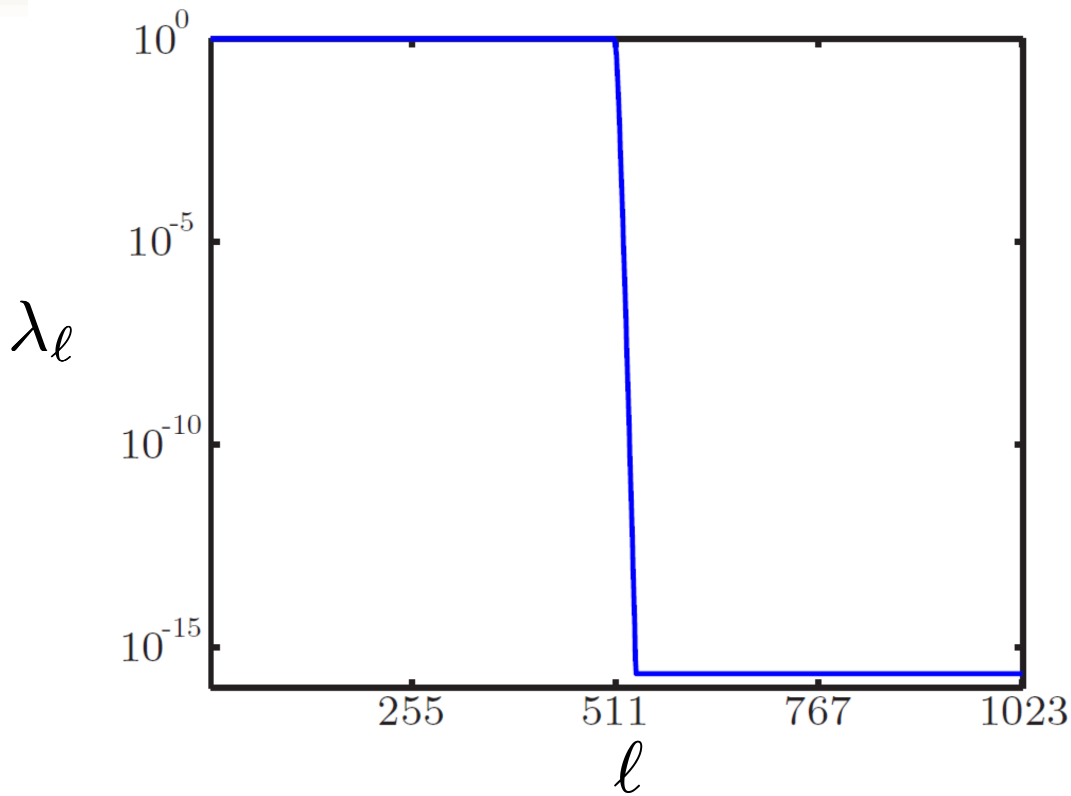
that satisfy

$$\mathcal{T}_N(\mathcal{B}_W(s_\ell))) = \lambda_\ell s_\ell.$$



The DPSS's are perfectly time-limited, but when $\lambda_\ell \approx 1$ they are highly concentrated in frequency.

DPSS Eigenvalue Concentration



$N = 1024$

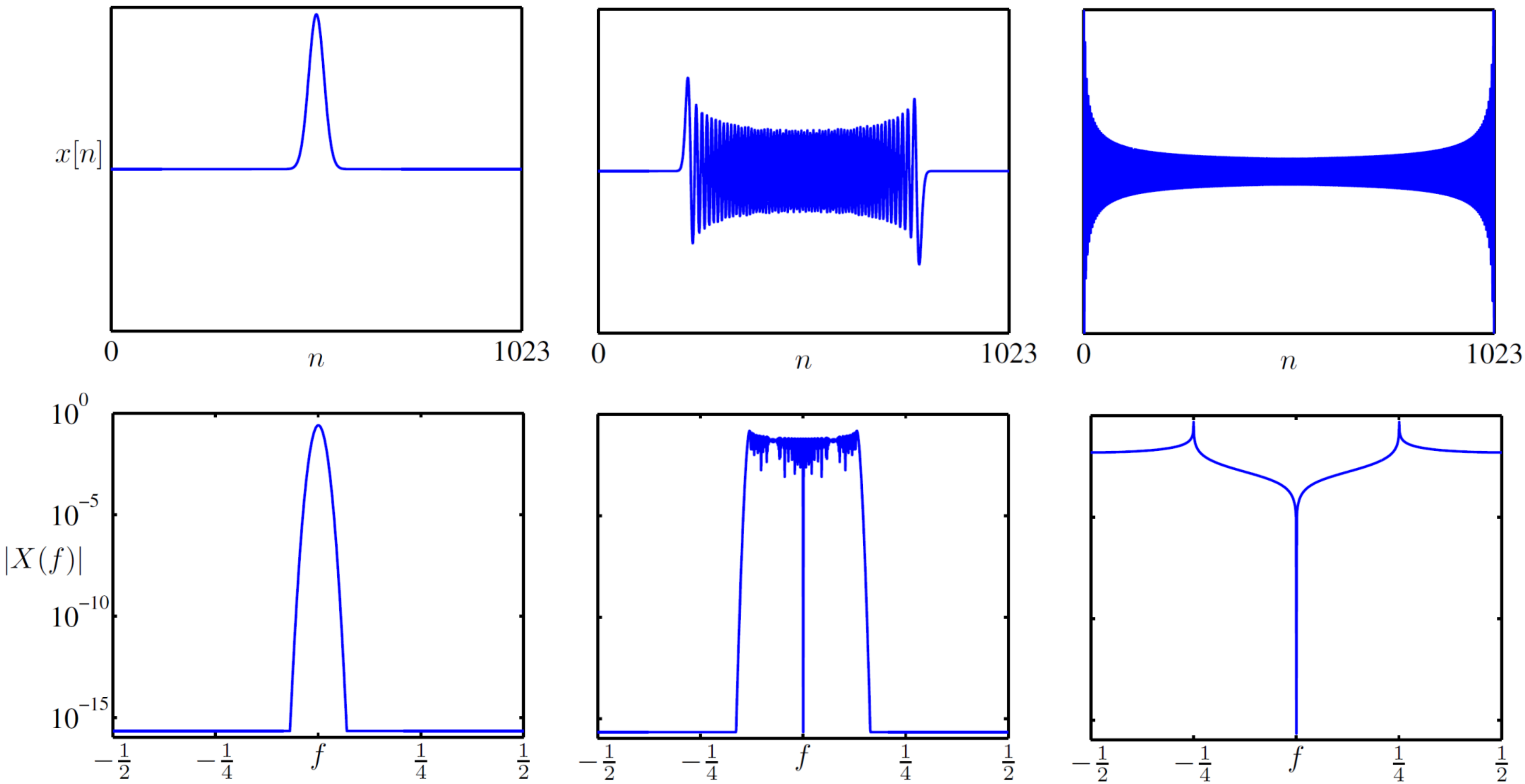
$W = \frac{1}{4}$

$2NW = 512$

The first $\approx 2NW$ eigenvalues ≈ 1.
The remaining eigenvalues ≈ 0.

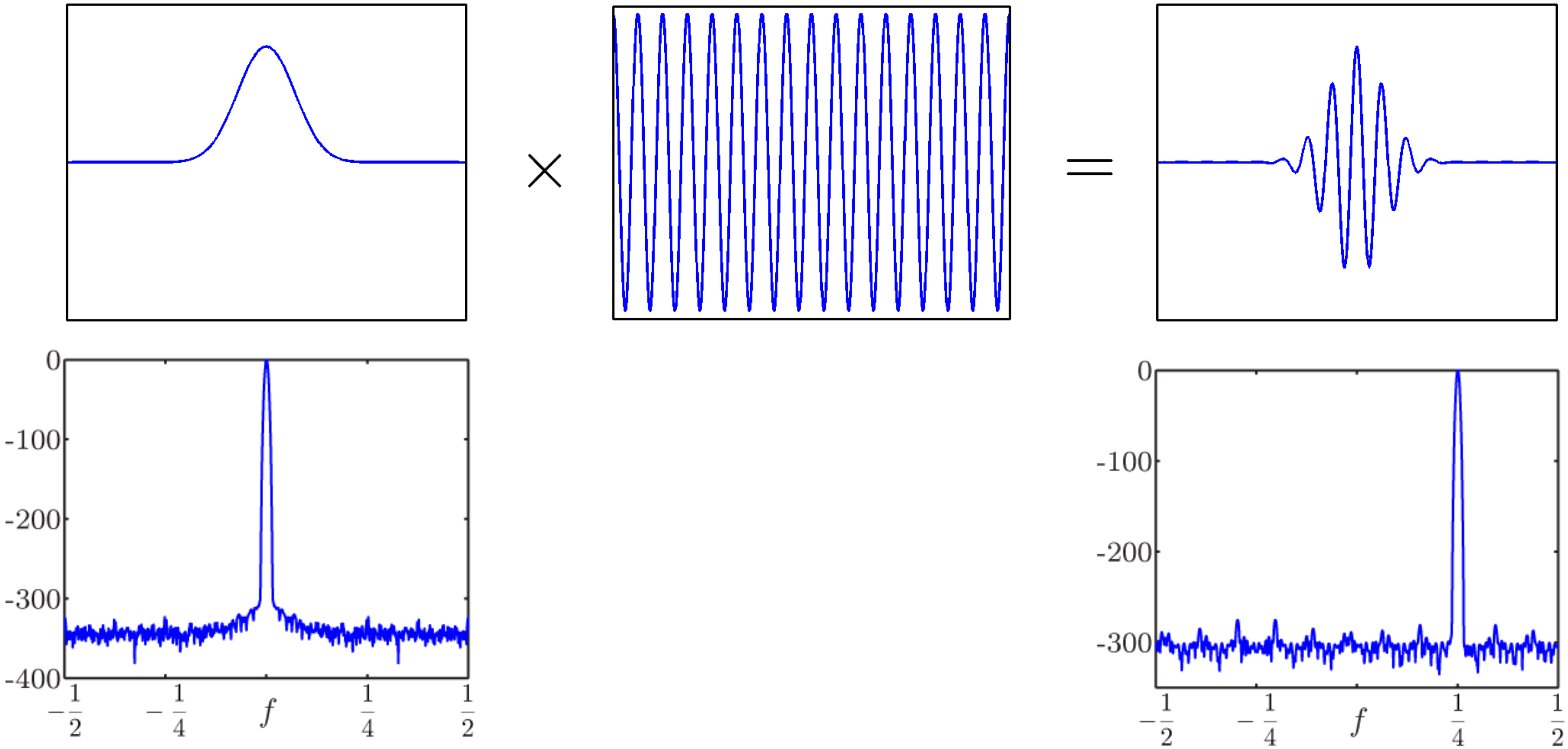
DPSS Examples

$$N = 1024 \qquad W = \frac{1}{4}$$



$\ell = 0$ $\qquad \ell = 127$ $\qquad \ell = 511$

DPSS's for Bandpass Signals

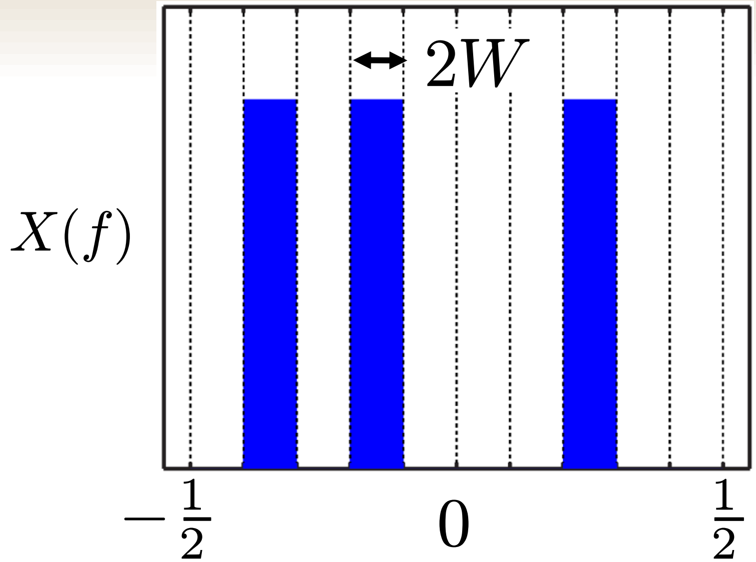


DPSS Dictionaries for CS

Modulate k DPSS vectors to center of each band:

$$\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_J]$$

approximately square if $k \approx 2NW$



J possible bands

Most multiband signals, when sampled and time-limited, are well-approximated by a sparse representation in Ψ .

Block-Sparse Recovery

Nonzero coefficients of α should be clustered in blocks according to the occupied frequency bands

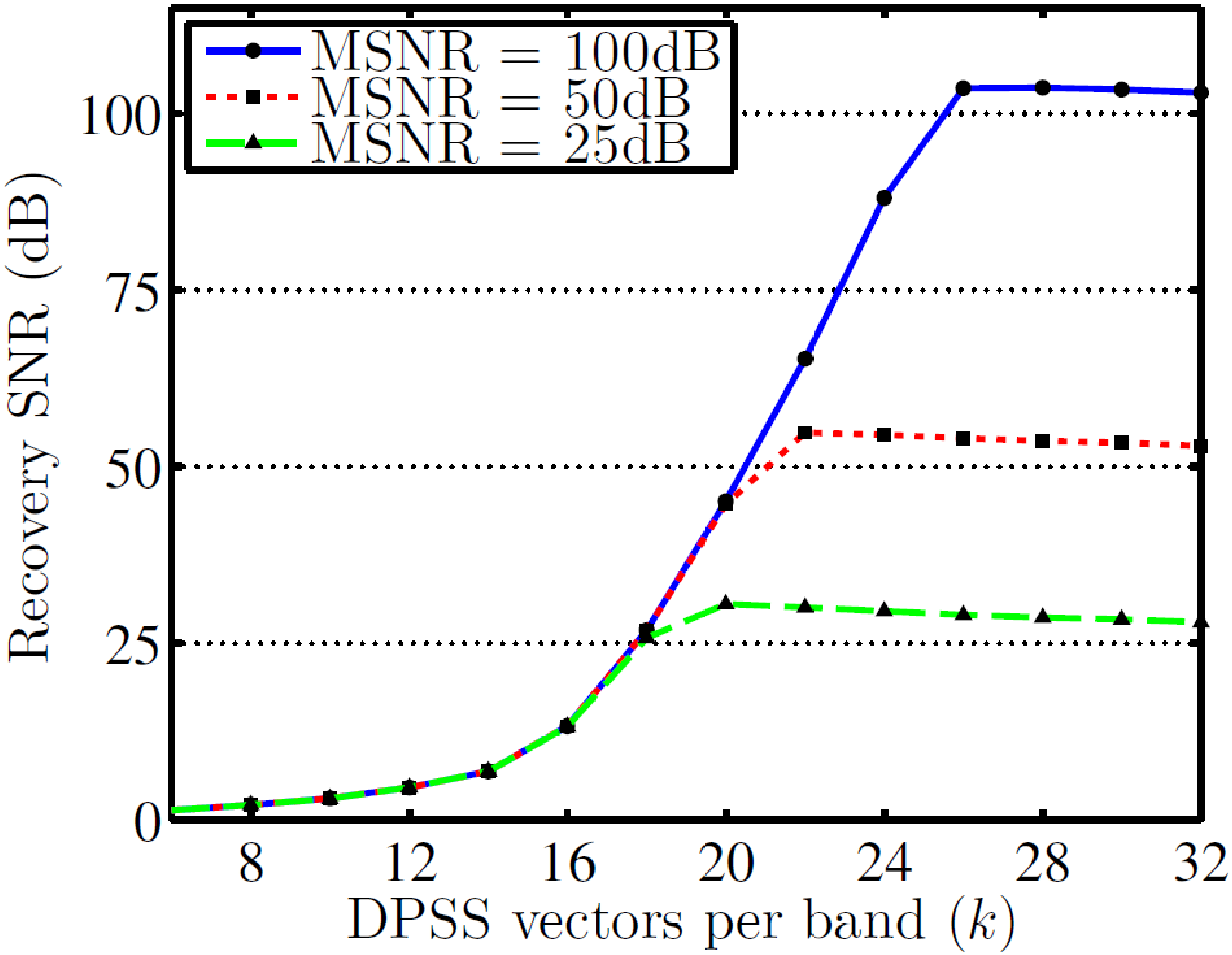
$$x = [\Psi_1, \Psi_2, \ldots, \Psi_J] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_J \end{bmatrix}$$

This can be leveraged to reduce the required number of measurements and improve performance through "model-based CS"

– Baraniuk et al. [2008], Blumensath and Davies [2009, 2011]

– Group LASSO

Empirical Results: Noise



$N = 4096$

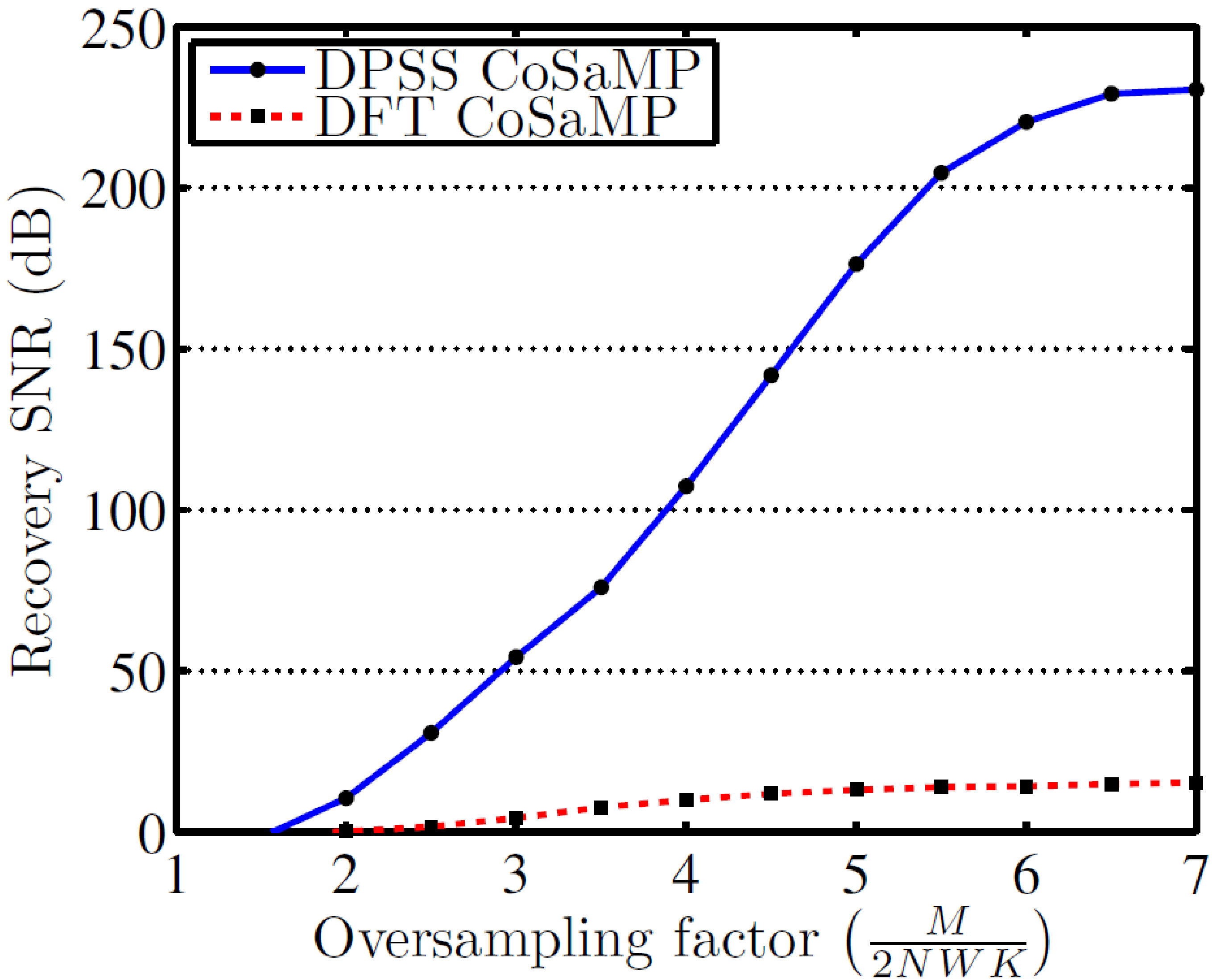
$M = 512$

$K = 5$

$\frac{B}{B_{\mathrm{nyq}}} = \frac{1}{256}$

[Davenport and Wakin - 2012]

Empirical Results: DFT Comparison



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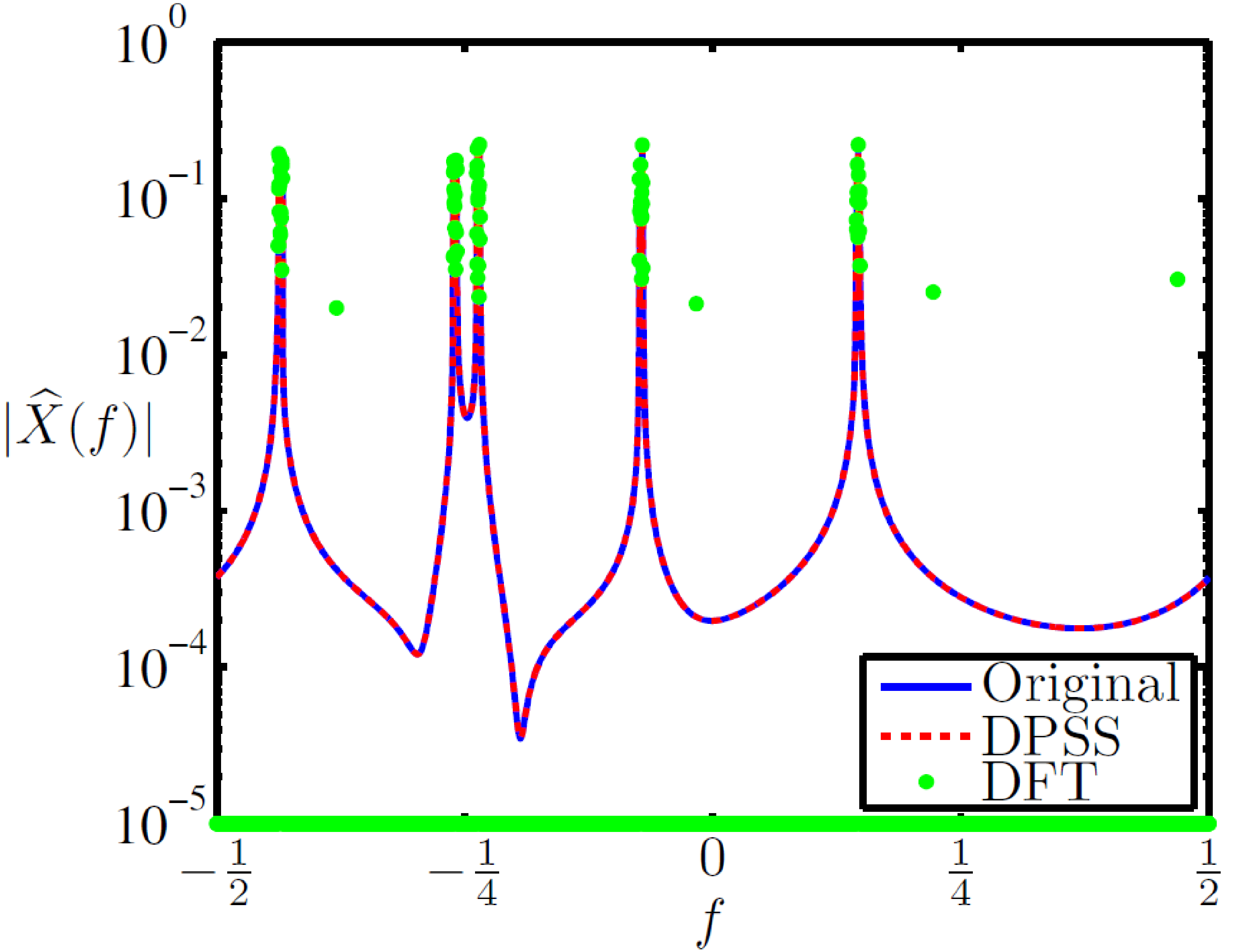

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$$\frac{B}{B_{\text{nyq}}} = \frac{1}{256}$$

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[Davenport and Wakin - 2012]

Conclusions

- DPSS's can be used to efficiently represent *most* sampled multiband signals
  - far superior to DFT

- Two types of error: *approximation* + *reconstruction*
  - approximation: small for most signals
  - reconstruction: tends to be small
  - delicate balance in practice, seems to be a sweet spot

- This approach combines careful design of Ψ with more sophisticated sparse models
  - relevant in many contexts beyond ADCs