Compressive Sensing
Part II: Sensing Matrices and Real-World Sensors

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Compressive Sensing

Replace samples with general *linear measurements*

\[ y = \Phi x \]

- **$M \times 1$** measurements
- **$M \times N$** sampled signal
- **$N \times 1$** $S$-sparse

[Donoho; Candès, Romberg, and Tao - 2004]
Analog Sensing is Matrix Multiplication

If $x(t)$ is bandlimited,

$$y[m] = \langle \phi_m(t), x(t) \rangle = \sum_{n=-\infty}^{\infty} x[n] \langle \phi_m(t), \text{sinc}(t/T_s - n) \rangle$$

$$\begin{bmatrix} y \\ \vdots \end{bmatrix} = \begin{bmatrix} \Phi & \cdots & \Phi \end{bmatrix} \begin{bmatrix} x \\ \vdots \end{bmatrix}$$

$y$ is an $M \times 1$ vector

$\Phi$ is an $M \times N$ matrix

$x$ is an $N \times 1$ vector

Nyquist-rate samples of $x(t)$
Sensing Matrix Design
Restricted Isometry Property (RIP)

\[ 1 - \delta \leq \frac{\| \Phi x_1 - \Phi x_2 \|^2}{\| x_1 - x_2 \|^2} \leq 1 + \delta \quad \| x_1 \|_0, \| x_2 \|_0 \leq S \]

\[ 1 - \delta \leq \frac{\| \Phi x \|^2}{\| x \|^2} \leq 1 + \delta \quad \| x \|_0 \leq 2S \]
RIP and Stability

If we want to guarantee that

$$\|x - \hat{x}\|_2 \leq C \|e\|_2$$

then we must have

$$\frac{1}{C} \leq \frac{\|\Phi x\|_2^2}{\|x\|_2^2} \quad \|x\|_0 \leq 2S$$
How Many Measurements?

If $\Phi$ satisfies the RIP with constant $\delta$, then

$$M > C_{S,\delta} S \log (N/S)$$

Sketch of proof: Construct a set $\mathcal{X}$ such that

- for any $x \in \mathcal{X}$, $\|x\|_0 = S$
- $|\mathcal{X}| \approx (N/S)^S$
- for any pair $x, y \in \mathcal{X}$, $1 \leq \|x - y\|_2 \leq 2$
Sub-Gaussian Distributions

- **Sub-Gaussian**: \( \mathbb{E} (e^{Xt}) \leq e^{c^2 t^2 / 2} \)
  - Gaussian
  - Bernoulli/Rademacher (±1)
  - any bounded distribution

- **Strictly sub-Gaussian**: \( \mathbb{E} (e^{Xt}) \leq e^{\sigma^2 t^2 / 2} \)

- For any \( x \), if the entries of \( \Phi \) are sub-Gaussian, then there exist \( \alpha \) and \( \beta \) such that w.h.p.

\[
\alpha \|x\|_2^2 \leq \|\Phi x\|_2^2 \leq \beta \|x\|_2^2
\]

**Strictly sub-Gaussian**

\[ \alpha = 1 - \delta, \quad \beta = 1 + \delta \]
Johnson-Lindenstrauss Lemma

- Stable projection of a discrete set of $\mathcal{P}$ points

- Pick $\Phi$ at *random* using a *sub-Gaussian* distribution

- For any fixed $x$, $\|\Phi x\|_2$ concentrates around $\|x\|_2$ with (exponentially) high probability

- We preserve the length of all $O(P^2)$ difference vectors simultaneously if $M = O(\log P^2) = O(\log P)$. 
JL Lemma Meets RIP

\[ 1 - \delta \leq \frac{\|\Phi x\|_2^2}{\|x\|_2^2} \leq 1 + \delta \] \[ \|x\|_0 \leq 2S \]

\[ P = O \left( \left( \frac{N}{S} \right)^S \right) \quad \Rightarrow \quad M = O \left( S \log \left( \frac{N}{S} \right) \right) \]

[Baraniuk, D, DeVore, and Wakin -2008]
RIP Matrix: Option 1

- Choose a *random matrix*
  - fill out the entries of $\Phi$ with i.i.d. samples from a sub-Gaussian distribution
  - project onto a “random subspace”

\[ M = O(S \log(N/S)) \ll N \]

[Baraniuk, D, DeVore, and Wakin -2008]
RIP Matrix: Option 2

- Random Fourier submatrix

\[ M = O(S \log^p (N/S)) \ll N \]

[Candès and Tao - 2006]
“Fast JL Transform”

- By first multiplying by random signs, a random Fourier submatrix can be used for efficient JL embeddings.

- If you multiply the columns of *any* RIP matrix by random signs, you get a JL embedding!

[Ailon and Chazelle - 2007; Krahmer and Ward - 2010]
Hallmarks of Random Measurements

**Stable**
With high probability, $\Phi$ will preserve information, be robust to noise

**Universal**
$\Phi$ will work with *any* fixed orthonormal basis (w.h.p.)

**Democratic**
Each measurement has “equal weight”
Compressive Sensors in Practice
Tomography in the Abstract

\[ r = x \cos \theta + y \sin \theta \]

\[ p_\theta(r_1) \]

\[ p_\theta(r) = \int \int f(x, y) \delta(x \cos \theta + y \sin \theta - r) \, dx \, dy \]
Fourier-Domain Interpretation

- Each projection gives us a “slice” of the 2D Fourier transform of the original image
- Similar ideas in MRI
- Traditional solution: Collect lots (and lots) of slices
“OK, Mrs. Dunn. We’ll slide you in there, scan your brain, and see if we can find out why you’ve been having these spells of claustrophobia.”
CS for MRI Reconstruction

- 256x256 MRA
- Backproj., 29.00dB
- Fourier sampling
  80 lines (M~0.28N)
- Min TV, 34.23dB [CR]
Multi-Slice Brain Imaging

- Scan reduction: x2.4
- Transform: wavelet
Pediatric MRI

Traditional MRI  

CS MRI  

4-8 x faster!

[Vasanawala, Alley, Hargreaves, Barth, Pauly, and Lustig - 2010]
“Single-Pixel Camera”

\[ y[m] = \sum_{n \in I_m} x[n] \]

\[ x[n] = \int_{\text{pixel}} \int_{\text{n}} x(t_1, t_2) \, dt_1 \, dt_2 \]

[Duarte, D, Takhar, Laska, Sun, Kelly, and Baraniuk - 2008]
1 Chip DLP™ Projection

- DLP Board
- Processor
- Memory
- Projection Lens
- DMD
- Shaping Lens
- Color Filter
- Condensing Lens
- Optics
- Light Source
- Screen
TI Digital Micromirror Device
“Single-Pixel” Camera

Object

LED (light source)

Lens 1

Lens 2

Photodiode circuit

DMD+ALP Board
“Single-Pixel” Camera

- Object
- LED (light source)
- Lens 1
- Lens 2
- Photodiode circuit
- DMD+ALP Board
“Single-Pixel” Camera

Object

LED (light source)

Photodiode circuit

Lens 2

Lens 1

DMD+ALP Board
“Single-Pixel” Camera
oops, crash, seven million years bad luck !?!

I can’t wait to take this on my next vacation

This is me skydiving

This is me swimming with dolphins

This is me at the grand canyon
First Images

Original 16384 Pixels 1600 Measurements (10%)

65536 Pixels 1300 Measurements (2%)

65536 Pixels 3300 Measurements (5%)
World’s First Photograph

- 1826, Joseph Niepce
- Farm buildings and sky
- 8 hour exposure
Color Imaging

Merging RGB channels

Two strategies:
1. Prism assembly
2. Layered sensors (ala Foveon)

Color Filter Wheel

4096 Pixels
800 (20%) Measurements

4096 Pixels
1600 (40%) Measurements

Foveon Image Array
“Single-Pixel” Camera

single photon detector

© MIT Tech Review
Low-Light Imaging with PMT

True color low-light imaging:
256 x 256 image with 10:1 compression
IR Imaging

1%  
2%  
5%  
10%  
100%
IR Imaging

Raster scans: Light from only one pixel

32 × 32

128 × 128

256 × 256

Compressive sensing:

Light from half the pixels

256 × 256
Hyperspectral Imaging

Sum of all bands

Real target
Hyperspectral Imaging
THz Imaging

Object mask

300 measurements

600 measurements

32 x 32 PCB masks

[Mittleman Group, Rice University]
THz Imaging: Sampling in Fourier

[Image of THz transmitter and receiver setup with diagrams of object position, metal aperture, and translation stage]

- Fourier Transform of object (Magnitude-only)
- CPR Reconstruction (4096 measurements)
- CSPR Reconstruction (1000 measurements)

[Mittleman Group, Rice University]
Compressive ADCs

DARPA “Analog-to-information” program:
Build high-rate ADC for signals with sparse spectra
Compressive ADCs

DARPA “Analog-to-information” program: Build high-rate ADC for signals with sparse spectra

[Le - 2005; Walden - 2008]
Analog-to-Information Conversion

- Many applications - particularly in RF - have hit an ADC performance *brick wall*
  - limited bandwidth (# Hz)
  - limited dynamic range (# bits)
  - deluge of bits to process downstream

- “Moore’s Law” for ADC’s: doubling in performance only every 6 years

- Inspiration from CS:
  - “analog-to-information” conversion
Random Demodulator

\[ x(t) \times p_c(t) \rightarrow \text{Integrator} \rightarrow \text{Sample-and-Hold} \rightarrow \text{Quantizer} \rightarrow y[n] \]

\[ X(f) \]

[Tropp, Laska, Duarte, Romberg, and Baraniuk - 2010]
Random Demodulator

\[ x(t) \times p_c(t) \rightarrow \text{Integrator} \rightarrow \text{Sample-and-Hold} \rightarrow \text{Quantizer} \rightarrow y[n] \]

\[ X(f) \]

[Tropp, Laska, Duarte, Romberg, and Baraniuk - 2010]
Random Demodulator

$x(t) \times p_c(t) \rightarrow \int \rightarrow \text{Sample-and-Hold} \rightarrow \text{Quantizer} \rightarrow y[n]$
Empirical Results

\[ 1.69S \log\left(\frac{N}{S} + 1\right) + 4.51 \]

\[ 1.71S \log\left(\frac{N}{S} + 1\right) + 1 \]

\[ M \approx 1.7S \log\left(\frac{N}{S} + 1\right) \]

[Tropp, Laska, Duarte, Romberg, and Baraniuk - 2010]
Example: Frequency Hopper

Nyquist rate sampling

20x sub-Nyquist sampling
Compressive Multiplexor

Random Demodulator

Compressive multiplexor is more digital

[Slavinsky, Laska, D, and Baraniuk - 2011]
Compressive Multiplexor in Hardware

• Boils down to:
  - 1 LFSR
  - $J$ switches
  - $2J$ resistors
  - 2 op amps
  - 1 low-rate ADC

1.1mm x 1.1mm ASIC on its way!

[Slavinsky, Laska, D, and Baraniuk - 2011]
Compressive ADCs: Challenges Ahead

- **Calibration!**
  - you must know $\Phi$ to recover (or do anything else)
  - big challenge for all approaches
  - can often be mitigated by certain design choices

- **Algorithms**
  - recovery algorithms are much faster than a few years ago, but still can’t operate in real time on GHz bandwidths
  - is recovery always necessary?

- **Applications**
  - noise can be a problem
  - good signal models are key
Compressive Sensors Wrap-up

- CS is built on a theory of *random measurements*
  - Gaussian, Bernoulli, random Fourier, fast JLT
  - stable, universal, democratic

- Randomness can often be built into real-world sensors
  - tomography
  - cameras
  - compressive ADCs
  - microscopes, sensor networks, DNA microarrays, radar, ...

- OK, we can build these devices. What are they actually good for? When are they appropriate?