Sketching with structured matrices for array imaging

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I. Coherent array imaging and sketching

We address a novel matrix sketching problem motivated by coherent imaging using antenna arrays. Sketching uses a suitable undetermined matrix to create a lower dimensional *sketch* of a linear system in problems like least squares regression and low rank approximation [1]. Research in this area has focused on reducing the time required to compute the sketch itself, as this can be as expensive as the original problem. The problem we address has a different flavor in that the sketching matrix has a particular structure and is motivated by a coherent imaging problem.

Coherent imaging in antenna arrays is performed by emitting signals onto a target and using the received reflections for recovery. The process of probing the target actively is done using *beamforming*, wherein specific combinations of the array inputs and outputs (beams) are emitted and measured [2, 3]. One of the bottlenecks in the imaging process is the number of beams required. Some techniques to overcome this bottleneck have been discussed in [4].

We present a novel trade-off between the number of beams and the bandwidth of the acquisition. In contrast to previous work on compressed sensing in array processing [5, 6, 7], we do not exploit any sparsity prior for the scene being imaged. Our model is simply that the scene is at a fixed, known range. In this case, if we measure the m array outputs independently or with a set of m beamforms, the scene can be captured by using a single frequency (dictated by the array geometry) and additional frequencies offer now new information. However, with a set of generic weights (*aperture codes*), longer wavelengths *do* reveal new information and this gives us the afore-mentioned trade-off. Our results revolve around finding conditions under which aperture coding *preserves the entire range* of the linear operator that maps the scene to the array measurements. Thus the results are independent of any model on the scene itself.

II. Problem formulation

To analyze this problem, we setup the aperture codes as a special sketching matrix on the imaging operator. If a scene is probed with k frequencies, at i^{th} frequency, the imaging process can be modeled as $y_i = A_i x_0$ where $x_0 \in \mathbb{R}^n$ is the target image, $y_i \in \mathbb{C}^m$ is the output of an m-element array and A_i is the linear operator mapping the scene to the measurements. For k frequencies, the entire process can be modeled as $y = [y_1^T\ y_2^T\ \cdots\ y_k^T]^T = [A_1^T\ A_2^T\ \cdots\ A_k^T]^T x_0 = Ax_0$. If a common set of l aperture codes are used at each frequency, with $l < m$, the aperture coded outputs can be expressed as

$$\Phi y = \Phi A x_0 = Y x_0 \tag{1}$$

where Φ is a block diagonal matrix with k repeated blocks, as shown below. We refer to such a matrix as an RBD (repeated block diagonal) matrix.

$$\Phi = \begin{bmatrix} \phi & 0 & \cdots & 0 \\ 0 & \phi & \cdots & 0 \\ 0 & 0 & \cdots & \phi \end{bmatrix} \tag{2}$$

We seek to establish that the least squares solutions of (1) and that of the original system $y = Ax_0$ are close.

III. Preliminary results

We recast this problem to that of capturing the subspace spanned by the significant right singular vectors of A using Φ. Similar to [8], we seek to establish a bound on $\|(I - P_{Y^*})A^*\|$ where P_{Y^*} is the projection onto the range of Y^*. To simplify the analysis, we assume that A has an exact rank r and that each A_i has rank r_i. Results similar to [8] can be used to show that if $l > \max_i(r_i)$, then $\|(I - P_{Y^*})A^*\| = 0$. We intend to show that l can be much smaller. To do this, we decompose A as below

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ A_k \end{bmatrix} = \begin{bmatrix} 0 & C_{11} & 0 & \cdots & 0 \\ 0 & C_{21} & C_{22} & \cdots & 0 \\ 0 & \cdot & \cdot & \cdots & \cdot \\ 0 & C_{k1} & C_{k2} & \cdots & C_{kk} \end{bmatrix} \begin{bmatrix} V_0^T \\ V_1^T \\ V_2^T \\ \cdot \\ V_k^T \end{bmatrix} = CV^T \tag{3}$$

where $C_{ij} \in \mathbb{R}^{m \times d_j}$ and each C_{ii} is full column rank when $d_i \neq 0$. V is an $n \times n$ orthonormal matrix and is such that $[V_1\ V_2\ \cdots V_i]$ includes an orthobasis for the row space of A_i. Intuitively, d_i's represent the dimension of the subspace of $row(A_i)$ not included in the row space of the previous row groups. Our main result is:

Theorem 1. *For a matrix A with d_i's defined as in* (3)*, the number of random vectors per block l needed to capture row space of A using an RBD matrix Φ with standard Gaussian random blocks is such that*

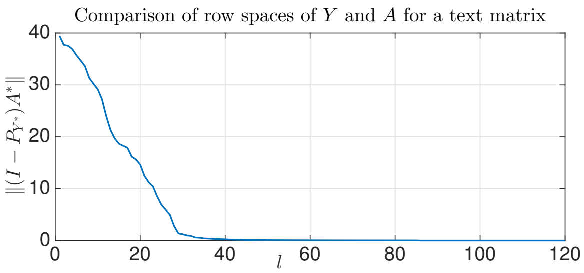
$$l \leq \max_i d_i = d_0 \tag{4}$$

This bound is tight when the row groups A_i span orthogonal subspaces ($C_{ij} = 0\ \forall i \neq j$). In other cases, l can be much smaller.

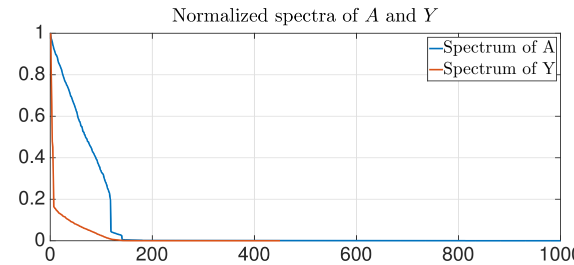
Although our result concerns exactly low-rank matrices, it can be generalized to matrices which are approximately low rank using perturbation theory for projection matrices [9]. In general, for matrices with numerical ranks much smaller than the ambient dimensions, the subspaces spanned by the A_i's will overlap. This overlap reduces the number of random vectors needed per block. This is especially relevant in the array imaging setting because the row groups are such that *row*(A_i) is approximately nested in *row*(A_j) $\forall i < j$.

Figure 1 shows the error $\|(I - P_{Y^*})A^*\|$ and the normalized spectra of the matrices A and ΦA for a test matrix of size 2000×1000, with $\max_i(r_i) \approx 300$ and $d_0 = 110$. The C_{ij}'s were filled with standard Gaussian random entries and the zero blocks in (3) were filled with Gaussian random entries with a variance of 10^{-4}. As the figures show, the row space was captured with $l \approx 45$.

In coherent imaging, y_i's are the samples of the 2D Fourier transform of the image in a trapezoidal area, as shown in Fig. 3. For our model, the array outputs at different frequencies are samples of the same function across nested intervals. Conventional beamforming uses a number of beams of the order of $\max_i(r_i)$. Hence, aperture coding can reduce the number of beams needed. This can be seen in Fig. 2 where simulations were performed for an array with 80×80 elements. While conventional method required about 800 beams, our method provides good reconstructions with just 50 randomized beams with 40 pulse frequencies between 2GHz and 4GHz.

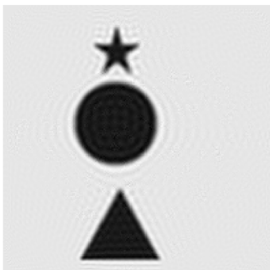


(a)

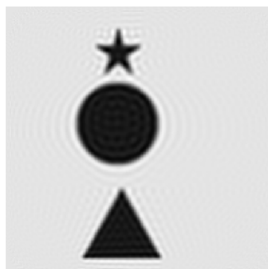


(b)

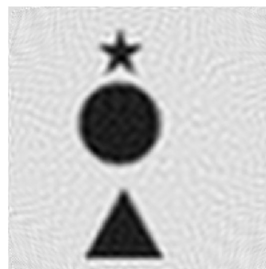
Fig. 1: (a) shows the error $\|(I - P_{Y^*})A^*\|$ and (b) shows the normalized spectrum of A and Y for $l = 45$. The number of significant singular values of A and Y can be seen to be approximately the same and hence they have approximately the same rank and row space.



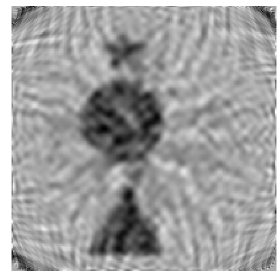
(a)



(b)



(c)



(d)

Fig. 2: A target scene reconstructed using aperture codes (a) shows the reconstruction obatined using conventional methods, which need about 800 beams for the settings of the experiment. (b), (c), (d) show reconstructions obtained using 200, 50 and 32 aperture codes.

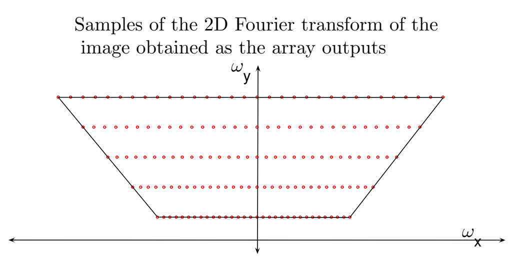


Fig. 3: The array outputs at different frequencies are the samples of the Fourier transform of the scene. The samples at the top row are obtained for the highest excitation frequency. For a scene at a fixed depth, the 2D Fourier transform is a function of only ω_x and hence we observe samples of the same function along nested intervals as we move from the lowest frequency to the highest.

References

[1] D. Woodruff, "Sketching as a tool for numerical linear algebra," *ArXiv preprint arXiv:1411.4357*, 2014.

[2] S. Patole and M. Torlak, "Two dimensional array imaging with beam steered data," *IEEE Trans. Image Process.*, vol. 22, no. 12, pp. 5181–5189, 2013.

[3] M. Soumekh, "Array imaging with beam-steered data," *IEEE Trans. Image Process.*, vol. 1, no. 3, pp. 379–390, 1992.

[4] J. Johnson, O. Oralkan, S. Ergun, U. Demirci, M. Karaman, and B. Khuri-Yakub, "Coherent array imaging using phased subarrays. Part II: Simulations and experimental results," *IEEE Trans. Ultrason., Ferroelect., Freq. Control*, vol. 52, no. 1, pp. 51–64, 2005.

[5] Y. Wiaux, L. Jacques, G. Puy, A. Scaife, and P. Vandergheynst, "Compressed sensing imaging techniques for radio interferometry," *Monthly Notices of the Royal Astronomical Society*, vol. 395, no. 3, pp. 1733–1742, 2009.

[6] N. Wagner, Y. Eldar, and Z. Friedman, "Compressed beamforming in ultrasound imaging," *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4643–4657, 2012.

[7] T. Chernyakova and Y. Eldar, "Fourier-domain beamforming: The path to compressed ultrasound imaging," *IEEE Trans. Ultrason., Ferroelect., Freq. Control*, vol. 61, no. 8, pp. 1252–1267, 2014.

[8] N. Halko, P. Martinsson, and J. Tropp, "Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions," *SIAM Review*, vol. 53, no. 2, pp. 217–288, 2011.

[9] B. Li, W. Li, and L. Cui, "New bounds for perturbation of the orthogonal projection," *Calcolo*, vol. 50, no. 1, pp. 69–78, 2013.