ECE 6270, Spring 2021

Homework #2

Due Sunday, Feb 7, at 11:59pm Suggested Reading: B&V, Sections 2.1-2.4 and 3.1-3.2. You might also want to skim Appendix A.

1. Prepare a one paragraph summary of what we talked about in the last week of class. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other classes?). The more insight you give, the better.

2. Provide feedback to your peers on Homework #1 in Canvas.

3. A function $f(\boldsymbol{x}) : \mathbb{R}^N \to \mathbb{R}$ is *concave* if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$,
$$f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \;\geq\; \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y}), \quad \text{for all} \;\; 0 \leq \theta \leq 1.$$
Give a simple yet rigorous argument that
$$f(\boldsymbol{x}) \text{ is concave} \quad \Leftrightarrow \quad -f(\boldsymbol{x}) \text{ is convex.}$$

4. Recall that a norm is a function $\|\cdot\| : \mathbb{R}^N \to \mathbb{R}$ which obeys
   - $\|\boldsymbol{x}\| \geq 0$ and $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.
   - $\|a\boldsymbol{x}\| = |a| \, \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^N$ and scalars $a \in \mathbb{R}$.
   - $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$.

   (a) Suppose that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$, where $\|\cdot\|$ denotes any valid norm. Prove that $f(\boldsymbol{x})$ is convex (using only the properties above).

   (b) Can $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$ ever be *strictly* convex? If so, give an example of such a norm. If not, provide a proof that no norm can be strictly convex.

5. The α-sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set $S_\alpha = \{x \;:\; f(x) \leq \alpha\}$.

   (a) Suppose f is convex. Show that S_α is convex for all $\alpha \in \mathbb{R}$.

   (b) Suppose f is convex. Show that the set of global minimizers of f is a convex set.

   (c) Recall that the unit ball of a norm is the set $\mathcal{B} = \{\boldsymbol{x} : \|\boldsymbol{x}\| \leq 1\}$. Show that the unit ball of any norm must be convex.

   (d) *Optional:* Suppose S_α is convex for all $\alpha \in \mathbb{R}$. Is f convex? Prove or find a counterexample.

6. (a) Let $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ be convex functions on $\mathbb{R}^N$. Show that
$$f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$$
is convex.

(b) Illustrate the above in $\mathbb{R}^1$ by making a sketch with affine functions $f_1(x) = a_1x + b_1$ and $f_2(x) = a_2x + b_2$. You may choose a_1, b_1, a_2, b_2 to your liking.

(c) Is it necessarily true that

$$f(\boldsymbol{x}) = \min\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}, \quad f_1, f_2 \text{ convex},$$

is convex? Sketch an example in $\mathbb{R}^1$ that supports your argument.

7. Recall that we use $\mathbb{S}_+^N$ to denote the set of $N \times N$ matrices that are symmetric and whose eigenvalues are non-negative.

(a) Consider the function

$$f(\boldsymbol{X}) = \min_{\boldsymbol{v}:\|\boldsymbol{v}\|_2=1} \boldsymbol{v}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{v}.$$

Note that f maps any arbitrary (not necessarily symmetric) matrix to a single real number. Show that $f(\boldsymbol{X})$ is concave.

(b) Use your result from the previous part to show that $\mathbb{S}_+^N$ is a convex set.

(c) Find a set of convex functions $f_1(\boldsymbol{X}), \dots, f_M(\boldsymbol{X})$ that map *arbitrary* $N \times N$ matrices to scalars ($f_m(\boldsymbol{X}) : \mathbb{R}^{N\times N} \to \mathbb{R}$) and scalars $b_1, \dots, b_M$ that specify $\mathbb{S}_+^N$, meaning

$$\boldsymbol{X} \in \mathbb{S}_+^N \quad \Leftrightarrow \quad f_m(\boldsymbol{X}) \le b_m, \text{ for all } m = 1, \dots, M.$$

(Note that if f_m is linear, then f_m is both convex and concave, and so $f_m(\boldsymbol{X}) = b_m$ can be implemented using the pair of inequalities $f_m(\boldsymbol{X}) \le b_m$ and $-f_m(\boldsymbol{X}) \le -b_m$.) Note that there are multiple valid answers to this problem.

8. Compute the first and second derivatives of the following functions (remember the product and chain rules).

(a) $f(x) = ax^2 + bx + c$, where a, b, c are constants.

(b) $f(x) = \sum_{m=1}^{M} \log(1 + e^{-a_m x})$, where $a_1, \dots, a_M$ are constants.

9. Compute the gradient and Hessian matrix of the following functions. Note that $\boldsymbol{x}$ is a vector in $\mathbb{R}^N$ in all the problems below.

(a) $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} + c$, where $\boldsymbol{A}$ is an $N \times N$ symmetric matrix (i.e., $\boldsymbol{A} = \boldsymbol{A}^T$), $\boldsymbol{b}$ is an $N \times 1$ vector, and c is a scalar.

(b) $f(\boldsymbol{x}) = \sum_{m=1}^{M} \log(1 + e^{-\boldsymbol{a}_m^T\boldsymbol{x}})$, where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_M$ are $N \times 1$ vectors.

10. Determine whether the following functions are convex, concave, or neither.

(a) $f(x) = e^{x^2}$ on $\mathbf{dom}\, f = \mathbb{R}$.

(b) $f(x) = \log(1 + e^x)$ on $\mathbf{dom}\, f = \mathbb{R}$.

(c) $f(x_1, x_2) = x_1x_2$ on $\mathbf{dom}\, f = \mathbb{R}_{++}^2$.

(d) $f(x_1, x_2) = 1/x_1x_2$ on $\mathbf{dom}\, f = \mathbb{R}_{++}^2$.

(e) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{dom}\, f = \mathbb{R}_{++}^2$.

(f) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{dom}\, f = \mathbb{R}_{++}^2$.