

GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Pre-test

Date: January 11, 2024

Course: ECE 6254

Name: _____

Last,

First

- Open book/internet.
- No time limit.
- The test is worth 100 points. There are ten questions, and each one is worth 10 points. In multi-part questions, each part will be weighted equally.
- **If an answer box is provided, please write your final answer in the box.**
- All work should be performed on the test itself. If more space is needed, use the backs of the pages.
- This test will be conducted under the rules and guidelines of the Georgia Tech Honor Code and no cheating will be tolerated (i.e., no discussing the test with other students).
- Make sure to look at the question titles, they will sometimes provide valuable hints.
- If you are unclear what a question is asking, please ask for clarification on Piazza. Please do not include your answer or ask if your approach is correct in any Piazza posts though.

Problem 1: Random variables. Suppose that three independent random variables X , Y , and Z are distributed according to

$$X \sim \text{Normal}(1, 1) \quad Y \sim \text{Normal}(1, 2) \quad Z \sim \text{Normal}(2, 4).$$

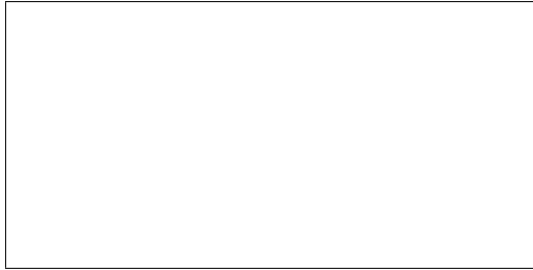
Here, $X \sim \text{Normal}(\mu, \sigma^2)$ denotes that X is a normal random variable with mean μ and variance σ^2 . What is the probability that $(X - Y)Z < 0$?

Problem 2: Independence: Suppose that the probability that A occurs is 0.8 and the probability that **both** A **and** B occur is 0.4. If A and B are independent events, what is the probability that **neither** A **nor** B occur?

Problem 3: Conditional probability density functions and derived distributions:
Suppose that X and Y have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) What are the marginal probability density functions for X and Y ?



(b) What is the probability density function for the random variable $R = \frac{X}{Y}$?



Problem 4: The median and the cumulative distribution function: Let M be the number of miles your electric car can drive without running out of electricity, and suppose that M has probability density function given by

$$f_M(m) = \begin{cases} \frac{e^{-m/481}}{481} & m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the median range for your car? That is, how far can you drive before there is a 50% chance that your battery runs out?

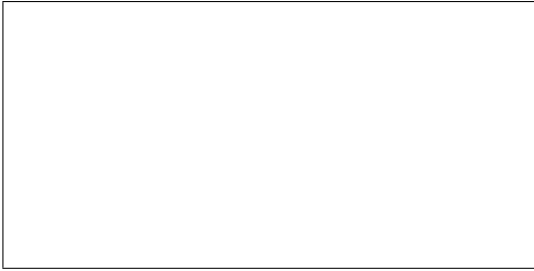
Problem 5: Bayes rule: Suppose that you have a newborn baby at home. Let X be the amount of time it takes (in hours) until your baby wakes up when you put her down, which we will model as an exponential random variable with parameter λ , i.e., X has probability density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately, you have no idea what λ is for your baby. You read online that a good model for λ is the pdf

$$f_\Lambda(\lambda) = \begin{cases} \frac{1}{\ln(2)\lambda} & \frac{1}{2} \leq \lambda \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that you put down your baby and she wakes up after $X = 2$ hours. Use this information to compute an updated pdf for Λ , i.e., $f_{\Lambda|X}(\lambda|2)$.



Problem 6: Joint probability density functions: The *correlation coefficient* $\rho(X, Y)$ between a pair of random variables X and Y is given by

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{\mathbb{E} \left[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y]) \right]}{\sqrt{\text{var}(X)\text{var}(Y)}}.$$

Let X and Y be independent random variables with $\text{var}(X) = 4$ and $\text{var}(Y) = 2$. We do not know $\mathbb{E}[X]$ or $\mathbb{E}[Y]$. Let $Z = 3X + Y$. What is the correlation coefficient $\rho(X, Z)$?

Problem 7: Pythagoras?

(a) Under what conditions on \mathbf{x} and \mathbf{y} is it true that

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 ?$$

(b) Under what conditions on \mathbf{x} and \mathbf{y} is it true that

$$\|\mathbf{x} + \mathbf{y}\|_2 = \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2 ?$$

Problem 8: Singular value decomposition.: Let

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & 2 & -2 & 0 \\ -2 & 2 & 2 & -2 & 0 \\ 2 & -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) What is $\text{rank}(\mathbf{A})$?

(b) Using Python or MATLAB (or whatever) find the singular value decomposition of \mathbf{A} . That is, find matrices $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$ such that

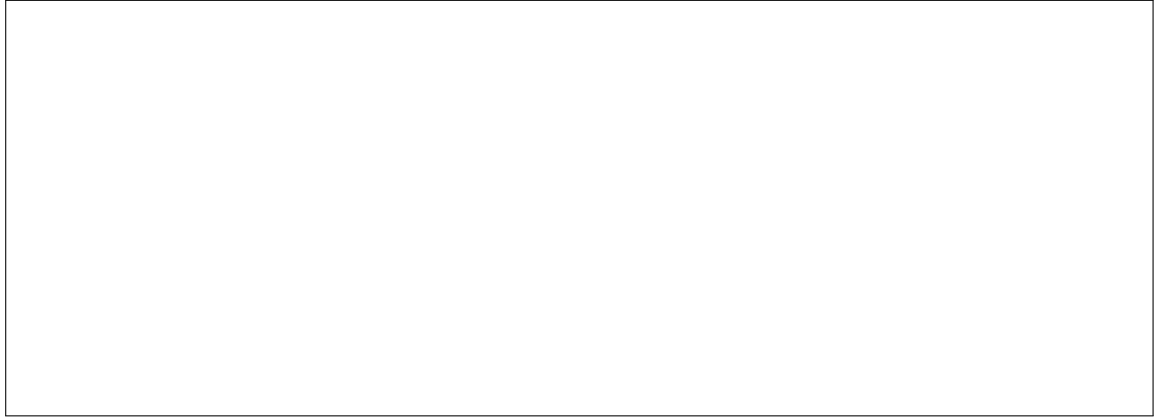
$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

and $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, $\mathbf{V}^T\mathbf{V} = \mathbf{I}$, and $\mathbf{\Sigma}$ has non-negative entries along its diagonal and is zero elsewhere.

(c) Describe, in words, the column space (or range) of \mathbf{A} :

$$\text{Range}(\mathbf{A}) = \{\mathbf{v} \in \mathbb{R}^5 : \mathbf{v} = \mathbf{A}\mathbf{x} \text{ for some } \mathbf{x}\}.$$

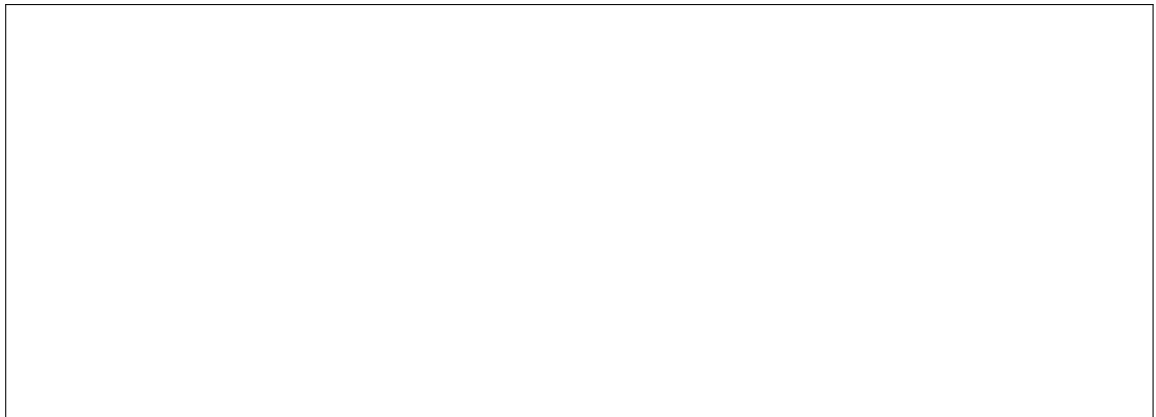
Note that I am not asking for a general definition of the column space of a matrix, but what the column space looks like for this specific \mathbf{A} .



(d) Describe, in words, the row space of \mathbf{A} (this is the column space of \mathbf{A}^T):

$$\text{Range}(\mathbf{A}^T) = \{\mathbf{v} \in \mathbb{R}^5 : \mathbf{v} = \mathbf{A}^T\mathbf{x} \text{ for some } \mathbf{x}\}.$$

Note that I am not asking for a general definition of the row space of a matrix, but what the row space looks like for this specific \mathbf{A} .



Problem 9: Eigenvalues and eigenvectors. Suppose that \mathbf{A} and \mathbf{B} are square symmetric matrices.

(a) Show that if \mathbf{A} and \mathbf{B} have the same eigenvectors, then they *commute*: $\mathbf{AB} = \mathbf{BA}$.

(b) Now suppose that both \mathbf{A} and \mathbf{B} are full rank. Show that if $\mathbf{AB} = \mathbf{BA}$ and \mathbf{A} has no repeated eigenvalues, then any eigenvector of \mathbf{A} is also an eigenvector of \mathbf{B} .

Problem 10: Orthogonal projections: Let

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 4 \\ -2 \\ -6 \\ -7 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix}.$$

Find a decomposition of \mathbf{x} into $\mathbf{x} = \mathbf{x}^* + \mathbf{x}_e$ where \mathbf{x}^* is in the span of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ – i.e., where $\mathbf{x}^* = \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 + \alpha_3 \mathbf{p}_3$ for some suitable choice of $\alpha_1, \alpha_2, \alpha_3$ – and where \mathbf{x}_e is orthogonal to the span of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. Make sure to give both \mathbf{x}^* and \mathbf{x}_e , and show your work/describe your method, even if you use a computer to help with the calculations.

Additional workspace:

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