Review of Fourier Transforms

The continuous-time Fourier transform (CTFT)

The CTFT of a signal $x(t)$ is

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt,$$

where $j = \sqrt{-1}$. The convention of using $j\Omega$ as the argument (instead of just Ω) is historical, and is common in the signal processing literature.

Anytime you see an integral expression like the one above, it is fair to ask whether or not it converges. If $x(t)$ is *absolutely integrable*, in that

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty,$$

then $X(j\Omega)$ is well-defined for all $\Omega \in \mathbb{R}$. It is also bounded, as in this case

$$|X(j\Omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \right| \leq \int_{-\infty}^{\infty} |x(t)|\, |e^{-j\Omega t}| dt = \int_{-\infty}^{\infty} |x(t)| dt.$$

If $x(t)$ has *finite energy*, in that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty,$$

then the Fourier transform is also well-defined, but you have to be a little more careful about what it means for two functions to be equal to one another. We will talk a little more about this later, but it is really just a mathematical detail which ends up not affecting our outlook on this topic at all.

The inverse CTFT is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega.$$

Parseval's theorem states that the energy in the time- and frequency-domains are equal to one another (to within a constant of $1/2\pi$):

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega.$$

The discrete-time Fourier transform (DTFT)

The DTFT of the sequence of numbers $\{x[n],\ n \in \mathbb{Z}\}$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$

Notes by J. Romberg and M. Davenport – August 18, 2019 – 21:07

Again, this sum is clearly well-defined (and bounded) when $x[n]$ is absolutely summable,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty,$$

and we can make sense of it when $x[n]$ has finite energy,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$

Notice that $X(e^{j\omega})$ is 2π-periodic, as

$$e^{-j\omega n} = e^{-j(\omega+2\pi\ell)n} \quad \text{for all} \quad \ell \in \mathbb{Z}.$$

The inverse DTFT is

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega.$$

The DTFT also preserves energy up to a constant, as

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

The Dirac delta function

The Dirac delta is a *generalized function*, defined through the relation

$$\int_{-L}^{L} x(t)\delta(t)dt = x(0), \quad \text{for any} \quad L > 0.$$

More generally,

$$\int_{t\in\mathcal{T}} x(t)\delta(t-t_0)dt = \begin{cases} x(t_0), & \text{if } t_0 \in \mathcal{T}, \\ 0, & \text{otherwise.} \end{cases}$$

$\delta(t)$ is not a function in the usual sense, but we can manipulate algebraically in much the same way we manipulate standard functions.

The delta function is the "derivative" of the Heaviside step function

$$\mu(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases},$$

in that they obey a relation of the same form as the Fundamental Theorem of Calculus:

$$\mu(t) = \int_{-\infty}^{t} \delta(\tau)\ d\tau.$$

The formalism for $\delta(t)$ and other generalized functions is found the mathematical theory of distributions. A nice overview of this theory can be found in the classic text *Distributions, Complex Variables, and Fourier Transforms*, by Hans Bremermann (1965).