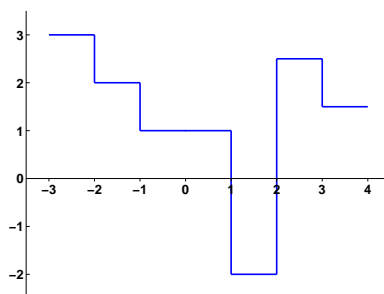


Haar wavelets

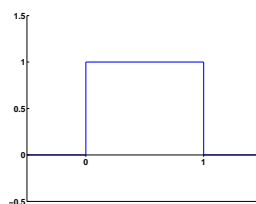
The Haar wavelet basis for $L_2(\mathbb{R})$ breaks down a signal by looking at the *difference* between piecewise constant approximations at different scales. It is the simplest example of a **wavelet transform**, and is very easy to understand.

Let \mathcal{V}_0 be the space of signals that are piecewise constant between the integers. Example member:



Set

$$\phi_0(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



It is clear that $\{\phi_0(t - n), n \in \mathbb{Z}\}$ is an orthobasis for \mathcal{V}_0 .

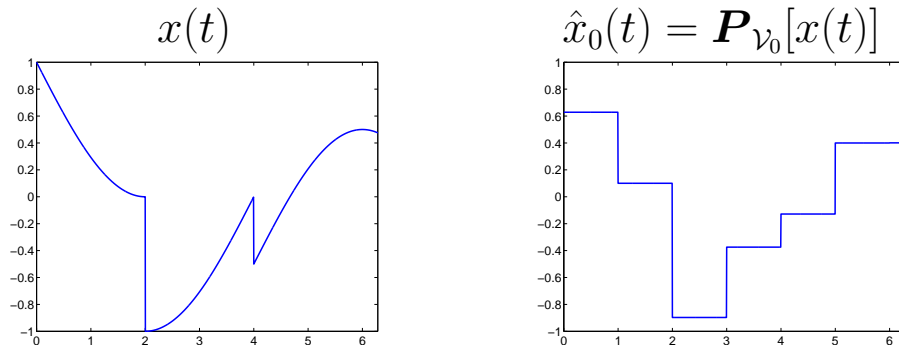
Given an arbitrary $x_0(t)$, we can find the best piecewise constant (between the integers) approximation by projecting onto the span of the $\{\phi_0(t - n)\}$. To keep the notation compact, we use

$$\phi_{0,n}(t) = \phi_0(t - n),$$

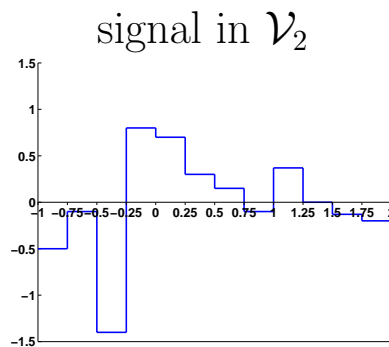
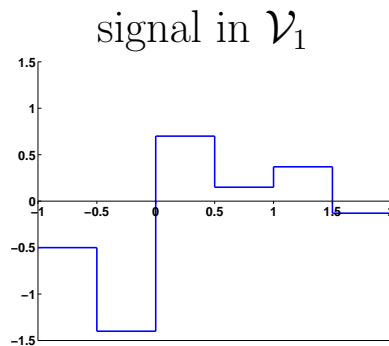
and then the closest signal in \mathcal{V}_0 to $x(t)$ is

$$\hat{x}_0(t) = \mathbf{P}_{\mathcal{V}_0}[x(t)] = \sum_{n=-\infty}^{\infty} \langle \mathbf{x}_0, \phi_{0,n} \rangle \phi_{0,n}(t).$$

Example:



We can do something similar for the space of piecewise constant functions at higher resolution. Let \mathcal{V}_j be the space of signals that are piecewise constant on intervals of length 2^{-j} .



We can get an orthobasis for \mathcal{V}_j simply by **contracting** the basis for \mathcal{V}_0 by a dyadic factor. Set

$$\phi_j(t) = 2^{j/2} \phi_0(2^j t),$$

and

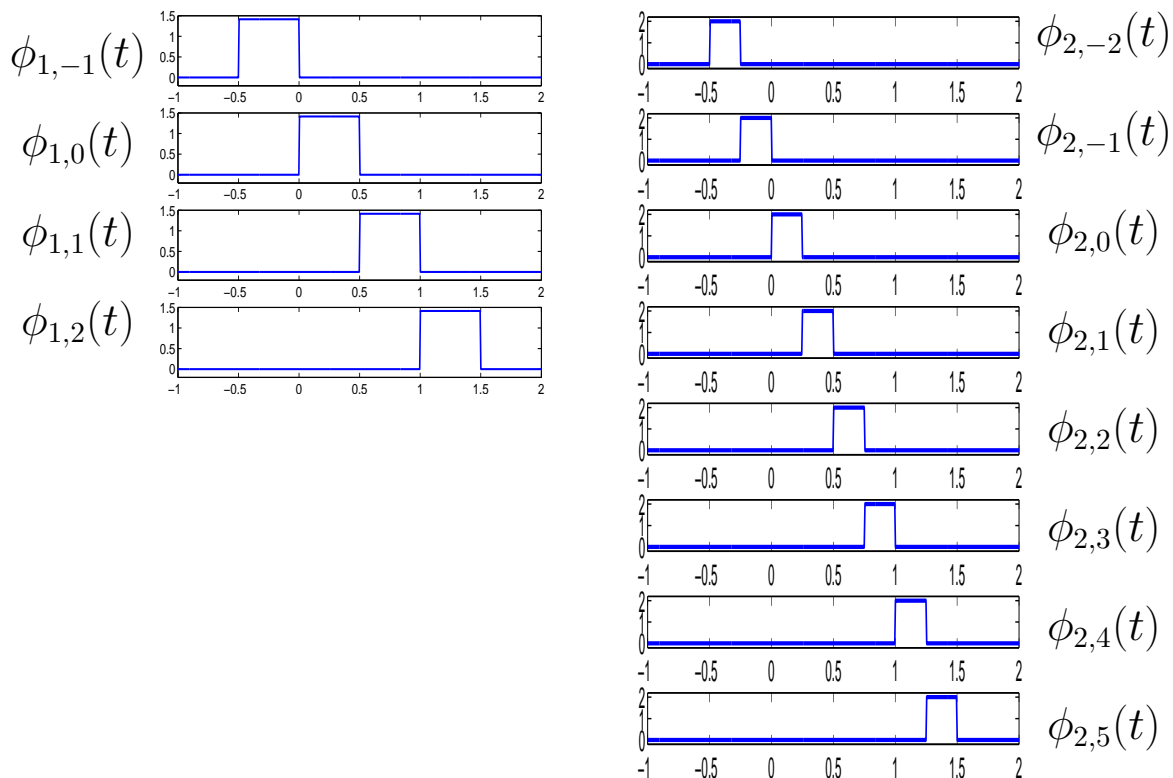
$$\phi_{j,n}(t) = \phi_j(t - 2^{-j}n) = 2^{j/2} \phi_0(2^j t - n).$$

Then it is easy to see that

$$\{\phi_{j,n}(t), n \in \mathbb{Z}\}$$

is an orthobasis for \mathcal{V}_j .

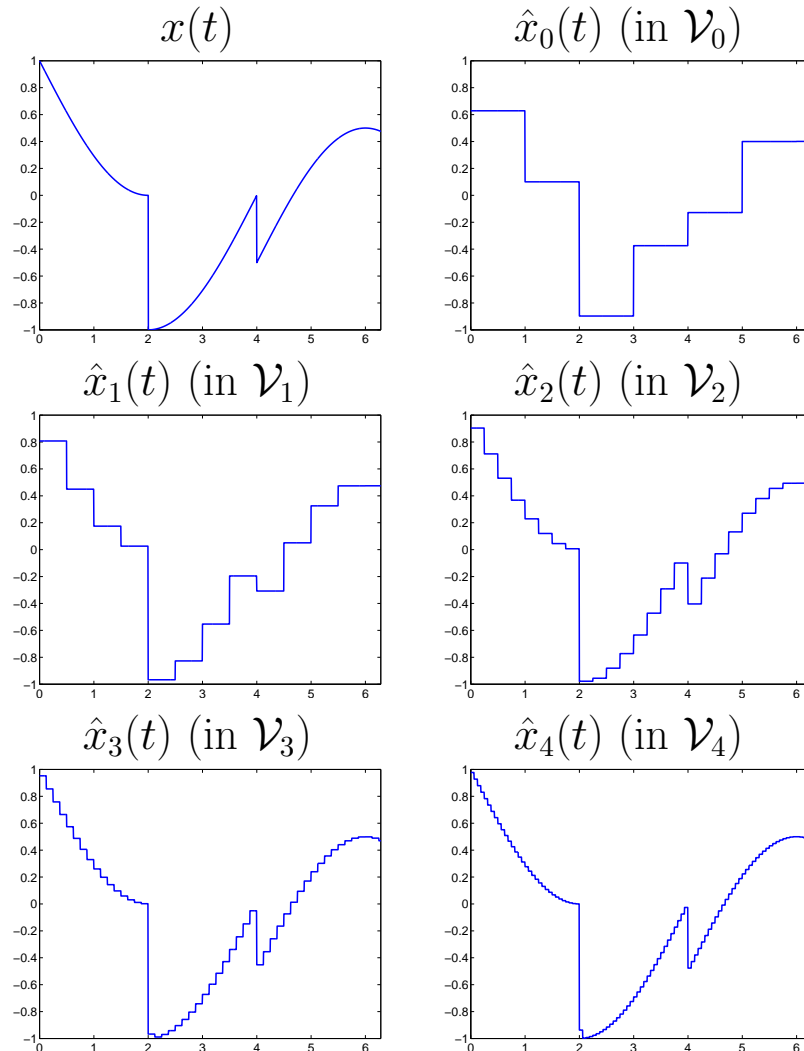
Examples:



The best approximation to an arbitrary signal $x(t) \in L_2(\mathbb{R})$ is again given by

$$\hat{x}_j(t) = \mathbf{P}_{\mathcal{V}_j}[x(t)] = \sum_{n=-\infty}^{\infty} \langle \mathbf{x}, \phi_{j,n} \rangle \phi_{j,n}(t)$$

As j increases this approximation becomes better and better:



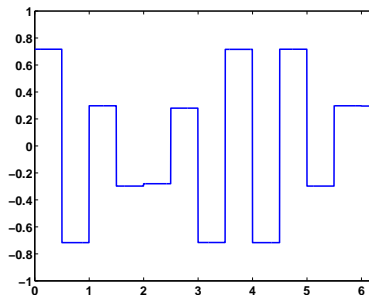
In the end, at “infinitely fine” resolution, we get the original signal back:

$$\lim_{j \rightarrow \infty} \hat{x}_j(t) = \lim_{j \rightarrow \infty} \mathbf{P}_{\mathcal{V}_j}[x(t)] = x(t).$$

The Haar wavelet basis represents the **differences** between these approximations. Set

$$\mathcal{W}_0 = \mathcal{V}_1 \ominus \mathcal{V}_0.$$

This is the space of signals that are in \mathcal{V}_1 but are orthogonal to everything in \mathcal{V}_0 . Signals in \mathcal{W}_0 are piecewise constant between the half integers (just like everything else in \mathcal{V}_1) but also have zero mean between the integers. Example member of \mathcal{W}_0 :

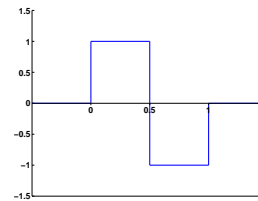


Similarly, we can set

$$\mathcal{W}_j = \mathcal{V}_{j+1} \ominus \mathcal{V}_j.$$

We can generate an orthobasis for the \mathcal{W}_j using the following function:

$$\psi_0(t) = \begin{cases} 1 & 0 \leq t < 1/2 \\ -1 & 1/2 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



With

$$\psi_j(t) = 2^{j/2}\psi(2^j t), \quad \psi_{j,n}(t) = \psi_j(t - 2^{-j}n) = 2^{j/2}\psi_0(2^j t - n),$$

then

$$\{\psi_{j,n}(t), n \in \mathbb{Z}\}$$

is an orthobasis for \mathcal{W}_j .

We now have the following multiscale decomposition for any $x(t)$:

$$x(t) = \underbrace{P_{\mathcal{V}_0}[x(t)] + P_{\mathcal{W}_0}[x(t)]}_{=P_{\mathcal{V}_1}[x(t)]} + P_{\mathcal{W}_1}[x(t)] + P_{\mathcal{W}_2}[x(t)] + \dots$$

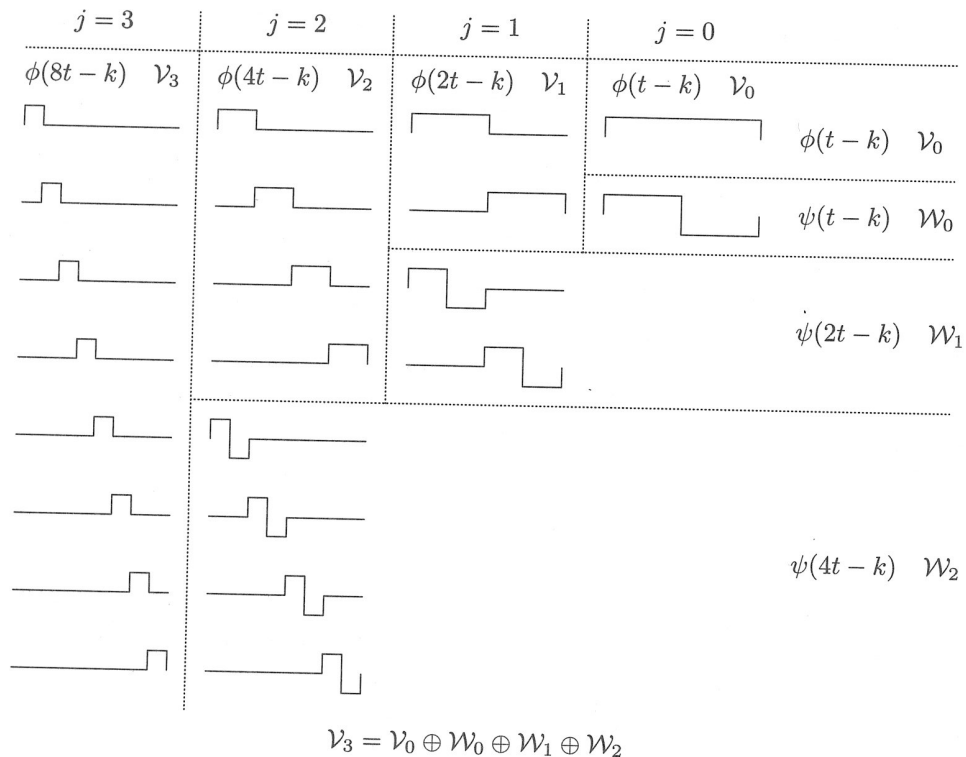
$$\underbrace{\hspace{10em}}_{=P_{\mathcal{V}_2}[x(t)]}$$

$$\underbrace{\hspace{15em}}_{=P_{\mathcal{V}_3}[x(t)]}$$

The reproducing formula for the **Haar wavelet system** is:

$$x(t) = \underbrace{\sum_{n=-\infty}^{\infty} \langle \mathbf{x}, \phi_{0,n} \rangle \phi_{0,n}(t)}_{=P_{\mathcal{V}_0}[x(t)]} + \underbrace{\sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} \langle \mathbf{x}, \psi_{j,n} \rangle \psi_{j,n}(t)}_{=P_{\mathcal{W}_j}[x(t)]}$$

Here are plots of the $\phi_{j,n}(t)$ and $\psi_{j,n}(t)$ at the first three scales:



(From Burrus et al, *Intro. to Wavelets and Wavelet Transforms*)

Nomenclature

- We call the \mathcal{V}_j the **scaling spaces** or **approximation spaces**.
- The $\{\phi_{j,n}, n \in \mathbb{Z}\}$ are an orthobasis for \mathcal{V}_j , and are called **scaling functions** at scale j .
- The \mathcal{W}_j are called **wavelet spaces** or **detail spaces**.
- The $\{\psi_{j,n}(t), n \in \mathbb{Z}\}$ are an orthobasis for \mathcal{W}_j , and are called **wavelets** at scale j .
- The expansion coefficients that specify the approximation \hat{x}_j in terms of the $\phi_{j,n}$ are called **scaling coefficients**: in the expression

$$\hat{x}_j = \sum_{n=-\infty}^{\infty} s_{j,n} \phi_{j,n}(t), \quad s_{j,n} = \langle \mathbf{x}, \phi_{j,n} \rangle,$$

the $s_{j,n}$ are scaling coefficients at scale j .

- The expansion coefficients for the projection onto \mathcal{W}_j are called **wavelet coefficients** $w_{j,n}$ at scale j :

$$w_{j,n} = \langle \mathbf{x}, \psi_{j,n} \rangle,$$

and

$$\mathbf{P}_{\mathcal{W}_j}[x(t)] = \sum_{n=-\infty}^{\infty} w_{j,n} \psi_{j,n}(t).$$

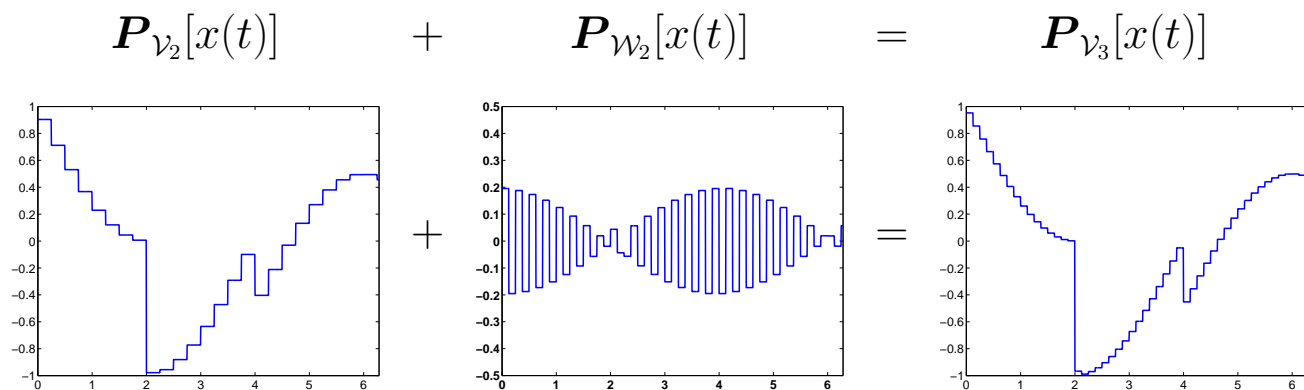
Moving between scales

We have seen that conceptually, the Haar transform analyzes a signal through a series of **multiscale approximations** onto nested subspaces \mathcal{V}_j , with

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset L_2(\mathbb{R}).$$

We can move between different approximation scales by adding in the “detail” spaces \mathcal{W}_j ,

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j.$$



At a fixed scale J , there are multiple ways to write the approximation \hat{x}_J in \mathcal{V}_J . The first is using just the scaling function at scale J :

$$\hat{x}_J(t) = \mathbf{P}_{\mathcal{V}_J}[x(t)] = \sum_{n=-\infty}^{\infty} s_{J,n} \phi_{J,n}(t).$$

Another way is to break up the approximation at two scales: approximate $x(t)$ in \mathcal{V}_{J-1} then add in details in \mathcal{W}_{J-1} to take you to the approximation in \mathcal{V}_J :

$$\begin{aligned} \hat{x}_J(t) &= \mathbf{P}_{\mathcal{V}_{J-1}}[x(t)] + \mathbf{P}_{\mathcal{W}_{J-1}}[x(t)] \\ &= \sum_{n=-\infty}^{\infty} s_{J-1,n} \phi_{J-1,n}(t) + \sum_{n=-\infty}^{\infty} w_{J-1,n} \psi_{J-1,n}(t) \end{aligned}$$

Yet another way is to break up the approximation at three scales: approximate $x(t)$ in \mathcal{V}_{J-2} , then add in the details in \mathcal{W}_{J-2} to take you to the approximation in \mathcal{V}_{J-1} , then add in the details in \mathcal{W}_{J-1} to take you to the approximation in \mathcal{V}_J :

$$\hat{x}_J(t) = \mathbf{P}_{\mathcal{V}_{J-2}}[x(t)] + \mathbf{P}_{\mathcal{W}_{J-2}}[x(t)] + \mathbf{P}_{\mathcal{W}_{J-1}}[x(t)].$$

Of course we could continue this all the way up to scale 0:

$$\begin{aligned} \hat{x}_J(t) &= \mathbf{P}_{\mathcal{V}_0}[x(t)] + \mathbf{P}_{\mathcal{W}_0}[x(t)] + \cdots + \mathbf{P}_{\mathcal{W}_{J-1}}[x(t)] \\ &= \sum_{n=-\infty}^{\infty} s_{0,n} \phi_{0,n}(t) + \sum_{j=0}^{J-1} \sum_{n=-\infty}^{\infty} w_{j,n} \psi_{j,n}(t). \end{aligned}$$

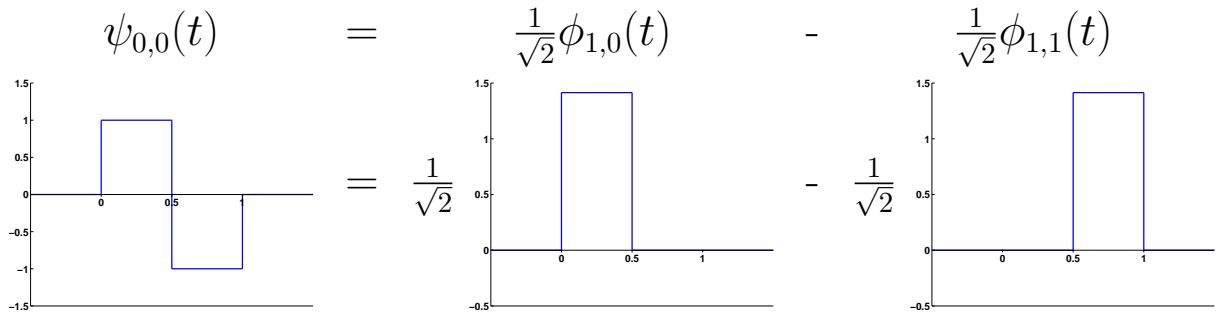
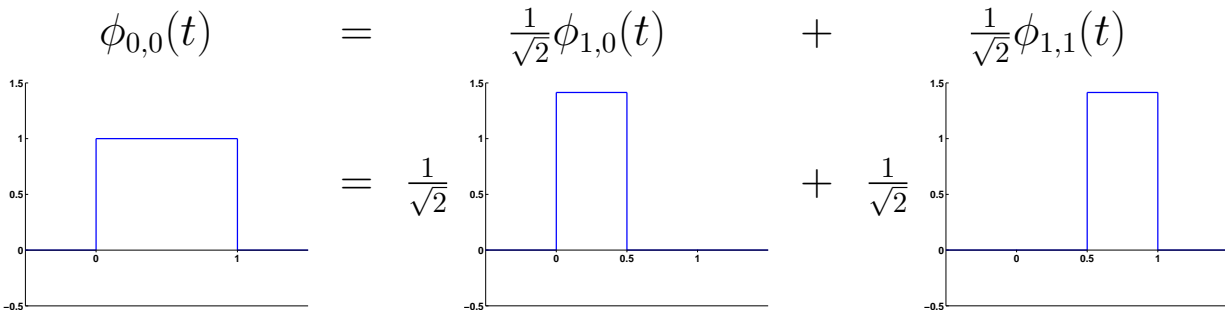
One of the key properties of the Haar wavelet transform (and all wavelet transforms, as we will see later) is that there is an efficient way to compute the multiscale approximation coefficients $\{s_{0,n}, w_{0,n}, w_{1,n}, \dots, w_{J-1,n}\}$ from the single scale approximation coefficients $\{s_{J,n}\}$.

What lets us do this is the simple fact that the scaling functions $\phi_{J-1,n}$ and wavelet functions $\psi_{J-1,n}$ at scale $J-1$ both can be built up out of scaling functions $\phi_{J,n}$ at scale J . To start, let us look at scale $\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0$. Now, as we illustrate below,

$$\begin{aligned} \phi_0(t) &= \frac{1}{\sqrt{2}} (\phi_1(t) + \phi_1(t - 1/2)), \\ &= \frac{1}{\sqrt{2}} (\phi_{1,0}(t) + \phi_{1,1}(t)), \end{aligned}$$

and

$$\psi_0(t) = \frac{1}{\sqrt{2}} (\phi_{1,0}(t) - \phi_{1,1}(t)).$$



We can generalize this relation to different shifts at scale 0:

$$\begin{aligned}\phi_{0,n}(t) &= \frac{1}{\sqrt{2}} (\phi_{1,2n}(t) + \phi_{1,2n+1}(t)), \\ \psi_{0,n}(t) &= \frac{1}{\sqrt{2}} (\phi_{1,2n}(t) - \phi_{1,2n+1}(t)),\end{aligned}$$

as well as between general scales j and $j + 1$:

$$\begin{aligned}\phi_{j,n}(t) &= \frac{1}{\sqrt{2}} (\phi_{j+1,2n}(t) + \phi_{j+1,2n+1}(t)), \\ \psi_{j,n}(t) &= \frac{1}{\sqrt{2}} (\phi_{j+1,2n}(t) - \phi_{j+1,2n+1}(t)).\end{aligned}$$

This immediately shows us how to compute the scaling coefficients $s_{j,n}$ and wavelet coefficients $w_{j,n}$ at scale j from the scaling coefficients $s_{j+1,n}$ at scale $j + 1$:

$$\begin{aligned} s_{j,n} &= \langle \mathbf{x}, \boldsymbol{\phi}_{j,n} \rangle \\ &= \frac{1}{\sqrt{2}} \langle \mathbf{x}, \boldsymbol{\phi}_{j+1,2n} \rangle + \frac{1}{\sqrt{2}} \langle \mathbf{x}, \boldsymbol{\phi}_{j+1,2n+1} \rangle \\ &= \frac{1}{\sqrt{2}} s_{j+1,2n} + \frac{1}{\sqrt{2}} s_{j+1,2n+1}, \end{aligned}$$

and

$$\begin{aligned} w_{j,n} &= \langle \mathbf{x}, \boldsymbol{\psi}_{j,n} \rangle \\ &= \frac{1}{\sqrt{2}} \langle \mathbf{x}, \boldsymbol{\phi}_{j+1,2n} \rangle - \frac{1}{\sqrt{2}} \langle \mathbf{x}, \boldsymbol{\phi}_{j+1,2n+1} \rangle \\ &= \frac{1}{\sqrt{2}} s_{j+1,2n} - \frac{1}{\sqrt{2}} s_{j+1,2n+1}. \end{aligned}$$

If we think of the scaling/wavelet coefficients at scale j as a discrete time sequence, so $s_j[n] := s_{j,n}$ and $w_j[n] := w_{j,n}$, then the expressions above suggest that the scaling coefficients at scale j can be broken down scaling and wavelet coefficients at scale $j - 1$ using **filters**. Next time we will discuss how wavelet transforms can be efficiently computed using **filterbanks**, which also provide another way to think about (and design) generalizations of Haar wavelets.