ECE 6250, Fall 2019

Homework #8

Due Wednesday November 6, at the beginning of class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. An $N \times N$ *circulant* matrix $\boldsymbol{H}$ has the form

$$\boldsymbol{H} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \tag{1}$$

   (a) Show that discrete Fourier vectors

$$\boldsymbol{f}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j2\pi k/N} \\ e^{j2\pi 2k/N} \\ \vdots \\ e^{j2\pi(N-1)k/N} \end{bmatrix}, \quad k = 0, \ldots, N-1$$

   are eigenvectors for every circulant matrix. What are the eigenvalues?

   (b) Explain, similar to what we did when discussing eigen-decompositions in the notes, why we can write

$$\boldsymbol{H} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^{\mathrm{H}} \tag{2}$$

   where $\boldsymbol{F}$ is the discrete Fourier matrix whose k^{th} column is $\boldsymbol{f}_k$ above ($\boldsymbol{F}^{\mathrm{H}}$ is the conjugate transpose of $\boldsymbol{F}$), and $\boldsymbol{D}$ is a diagonal matrix. Relate this to the fact that circular convolution in the time domain is multiplication in the DFT domain.

3. Suppose we have a machine that measures signals $f(t)$ in $L_2([0,1])$ by taking integrals over different regions:

$$y[m] = \int_{a_m}^{b_m} f(t)\ \mathrm{d}t, \quad 0 \le a_m \le b_m \le 1, \tag{3}$$

   for $m = 1, \ldots, M$. To recover $f(t)$, we model it as being a polynomial of order $N-1$,

$$f(t) = x_{N-1}t^{N-1} + x_{N-2}t^{N-2} + \cdots + x_1 t + x_0. \tag{4}$$

(a) Write a MATLAB function `intmat.m` that takes an M vector of lower limits `a`, a M vector of upper limits `b`, and a degree N and returns the $M \times N$ matrix `A` such that if $f(t)$ has the form (4), applying $\boldsymbol{A}$ to[1]

$$\boldsymbol{x} = \begin{bmatrix} x_{N-1} \\ x_{N-2} \\ \cdots \\ x_0 \end{bmatrix}$$

results in evaluations of the measurements in (3),

$$\boldsymbol{y} = \begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[m] \end{bmatrix} = \boldsymbol{A}\boldsymbol{x}.$$

(b) The file `hw8problem3.mat` contains vectors `a`, `b`, and `y` of length 20 representing a particular set of lower limits, upper limits, and associated measurements. Find the polynomial of order 9 ($N = 10$) that best describes these measurements in the least-squares sense. Plot your **synthesized estimate** $\hat{f}(t)$ as a function of time — that is, after estimating the coefficients $\hat{x}_{N-1}, \ldots, \hat{x}_0$, form

$$\hat{f}(t) = \hat{x}_{N-1}t^{N-1} + \cdots + \hat{x}_1 t + \hat{x}_0$$

and plot it as a function of time on $[0, 1]$.

4. (a) Let $\boldsymbol{A}$ be a $N \times N$ symmetric matrix. Show that[2]

$$\text{trace}(\boldsymbol{A}) = \sum_{n=1}^{N} \lambda_n,$$

where the $\{\lambda_n\}$ are the eigenvalues of $\boldsymbol{A}$.

(b) Recall the definition of the Frobenius norm of an $M \times N$ matrix:

$$\|\boldsymbol{A}\|_F = \left(\sum_{m=1}^{M} \sum_{n=1}^{N} |A[m,n]|^2 \right)^{1/2}.$$

Show that

$$\|\boldsymbol{A}\|_F^2 = \text{trace}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) = \sum_{r=1}^{R} \sigma_r^2,$$

where R is the rank of $\boldsymbol{A}$ and the $\{\sigma_r\}$ are the singular values of $\boldsymbol{A}$.

(c) The *operator norm* (sometimes called the *spectral norm*) of an $M \times N$ matrix is

$$\|\boldsymbol{A}\| = \max_{\boldsymbol{x} \in \mathbb{R}^N,\ \|\boldsymbol{x}\|_2 = 1} \|\boldsymbol{A}\boldsymbol{x}\|_2.$$

[1] The coefficients are ordered in $\boldsymbol{x}$ to match the MATLAB convention for polynomials.

[2] The trace of a matrix is the sum of the elements on the diagonal: $\text{trace}(\boldsymbol{A}) = \sum_{n=1}^{N} A[n,n]$.

(This matrix norm is so important, it doesn't even require a designation in its notation — if somebody says "matrix norm" and doesn't elaborate, this is what they mean.) Show that

$$\|\boldsymbol{A}\| = \sigma_1,$$

where σ_1 is the largest singular value of $\boldsymbol{A}$. For which $\boldsymbol{x}$ does

$$\|\boldsymbol{A}\boldsymbol{x}\|_2 = \|\boldsymbol{A}\| \cdot \|\boldsymbol{x}\|_2 \ \ ?$$

(d) Prove that $\|\boldsymbol{A}\| \leq \|\boldsymbol{A}\|_F$. Give an example of an $\boldsymbol{A}$ with $\|\boldsymbol{A}\| = \|\boldsymbol{A}\|_F$.

5. Suppose we have a signal $f(t)$ on $[0, 1]$ which is "bandlimited" in that it only has $N = 2B+1$ Fourier series coefficients which are non-zero:

$$f(t) = \sum_{k=-B}^{B} \alpha_k e^{j2\pi kt}, \tag{5}$$

for some set of expansion coefficents

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{-B} \\ \vdots \\ \alpha_0 \\ \vdots \\ \alpha_B \end{bmatrix} \in \mathbb{C}^N.$$

We observe samples M samples of $f(t)$ at locations $t_1, t_2, \ldots, t_M$ which are **not** necessarily uniformly spaced,

$$y[m] = f(t_m), \quad m = 1, \ldots, M. \tag{6}$$

(a) Write a MATLAB function `sampmat.m` that takes a vector `smptimes` of length M containing the sample locations and a dimension $N = 2B + 1$ (which you can assume is odd), and returns a $M \times N$ matrix $\boldsymbol{A}$ such that when $\boldsymbol{A}$ is applied to a vector of Fourier series coefficients (as in (5)), it returns the sample values in (6).

(b) The file `hw8problem5.mat` contains vectors `samptimes` and `y` of length $M = 259$, which contain sample times t_m and sample values $f(t_m)$. Find the signal of bandwidth $N = 51$ (so $B = 25$) that best explains these samples in the least-squares sense. Plot your **synthesized estimate** $\hat{f}(t)$ as a function of time.