ECE 6250, Fall 2019

Homework #6

Due Wednesday October 9, at the beginning of class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. (a) Let $b[n]$ be an FIR filter of length L. Show how

$$\sum_{n=1}^{L} n^q b[n] = 0 \quad \text{for all} \quad q = 0, \ldots, p \tag{1}$$

implies

$$b[n] \star x[n] = \sum_{k=-\infty}^{\infty} b[k]x[n-k] = 0,$$

when

$$x[n] = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0$$

for arbitrary $a_p, \ldots, a_0 \in \mathbb{R}$. That is, show that the discrete sequence $b[n]$ having p "vanishing moments" produces an output of zero when convolved with polynomials of order p.

(b) Now suppose we have a wavelet $\psi_0(t)$, which can be written as a superposition of scaling functions at scale $j = 1$ using

$$\psi_0(t) = \sum_n b[n]\phi_{1,n}(t).$$

Show that if the discrete *sequence* $b[n]$ has p vanishing moments (as in (1)), then the continuous time wavelet $\psi_0(t)$ must also have p vanishing moments, meaning

$$\int_{-\infty}^{\infty} t^q \psi_0(t) \ \mathrm{d}t = 0 \quad \text{for all} \quad q = 0, \ldots, p.$$

Note that the $\phi_{1,n}(t)$ will not in general have vanishing moments — just make the following constant substitutions when you see the integrals below:

$$C_0 = \int_{-\infty}^{\infty} \phi_{1,0}(t) \ \mathrm{d}t, \quad C_1 = \int_{-\infty}^{\infty} t\phi_{1,0}(t) \ \mathrm{d}t, \quad \cdots, \quad C_p = \int_{-\infty}^{\infty} t^p \phi_{1,0}(t) \ \mathrm{d}t.$$

(Hint: Start by showing this for $p = 0$, then $p = 1$, then generalize ... maybe by using the binomial theorem at some point.)

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3. Implement the Haar wavelet transform and its inverse in MATLAB. Do this by writing two MATLAB functions, `haar.m` and `ihaar.m` that are called as `w=haar(x,L)` and `x = ihaar(w,L)`. Here, `x` is the original signal, and `L` is the number of levels in the transform.

   You may assume that the length of `x` is dyadic; that is, the length of the input is 2^J for some positive integer J. In this situation, the Haar transform (no matter what L is) will have exactly 2^J terms, so the length of `x` and `w` should be the same.

   If we interpret the input `x` as being scaling coefficients at scale J, then the vector `w` should consist of the scaling coefficients at scale $J-L$ stacked on top of the wavelet coefficients for scale $J-L$ stacked on top of the wavelet coefficients for scale $J-L+1$, etc.

   Try your transform out on the data in `blocks.mat` and `bumps.mat`. For these two inputs, take a Haar wavelet transform with $L=3$ levels, and plot the scaling coefficients at scale $J-3$, and the wavelet coefficients at scales $J-3$ down to $J-1$. (For both of these signals, $J=10$.)

   Also, verify that your transform is energy preserving.

   Turn in printouts of your code along with the plots mentioned above.

4. (Optional) Consider the function[1]

$$\psi_0(t) = 2\cos\left(\frac{3\pi t}{2}\right)\frac{\sin(\pi t/2)}{\pi t},$$

   and let

$$\psi_{0,n}(t) = \psi_0(t-n).$$

   (a) Show that

$$\langle \boldsymbol{\psi}_{0,n}, \boldsymbol{\psi}_{0,k}\rangle \;=\; \int_{-\infty}^{\infty} \psi_{0,n}(t)\psi_{0,k}(t)\;\mathrm{d}t \;=\; \begin{cases} 1 & n=k \\ 0 & n\neq k \end{cases}.$$

   (b) What is $\operatorname{span}\left(\{\psi_{0,n}(t),\; n\in\mathbb{Z}\}\right)$? That is, for what space is $\{\boldsymbol{\psi}_{0,n}\}_{n\in\mathbb{Z}}$ an orthobasis?

   (c) Let

$$\psi_j(t) = 2^{j/2}\psi_0(2^j t) \quad \text{and} \quad \psi_{j,n}(t) = \psi_j(t-2^{-j}n) = 2^{j/2}\psi_0(2^j t - n)$$

   Notice that the $\{\psi_{j,n}(t)\}$ are dyadic shifts of $\psi_j(t)$ with spacing 2^{-j}. What is $\operatorname{span}\left(\{\psi_{j,n}(t),\; n\in\mathbb{Z}\}\right)$?

   (d) Let $\phi(t)$ be the standard sinc function

$$\phi(t) = \frac{\sin(\pi t)}{\pi t}, \quad \phi_n(t) = \phi(t-n).$$

   What is the span of the union of the $\{\phi_n(t)\}$ and all of the $\{\psi_{j,k}(t)\}_k$ for $j=0,1,\ldots,J$?

   (e) For a signal $x(t)$, how can we obtain $\langle \boldsymbol{x}, \boldsymbol{\psi}_{j,n}\rangle$ for all n using a filter-then-sample architecture?

[1] General hint for this question: Do everything in the Fourier domain. Start by sketching the continuous-time Fourier transform of $\psi_0(t)$ and $\psi_j(t)$ (when you get to part c).