ECE 6250, Fall 2019

Homework #4

Due Wednesday September 18, at the beginning of class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. The vector space $L_2([0,1]^2)$ is the space of signals of two variables, $x(s,t)$ with $s,t \in [0,1]$ such that
$$\int_0^1 \int_0^1 |x(s,t)|^2 \,\mathrm{d}s \,\mathrm{d}t \;<\; \infty.$$
Let $\{\psi_k(t),\ k \geq 0\}$ be an orthobasis for $L_2([0,1])$. Define
$$v_{k,\ell}(s,t) = \psi_k(s)\,\psi_\ell(t), \quad k,\ell \geq 0.$$
Show that $\{v_{k,\ell}(s,t),\ k,\ell \geq 0\}$ are an orthobasis for $L_2([0,1]^2)$.

3. Let $\mathcal{E}$ be the space of signals on $[-1,1]$ that are even:
$$x(t) \in \mathcal{E} \quad \Leftrightarrow \quad x(t) = x(-t), \quad t \in [-1,1],$$
and let $\mathcal{O}$ be the space of signals on $[-1,1]$ that are odd:
$$x(t) \in \mathcal{O} \quad \Leftrightarrow \quad x(t) = -x(-t), \quad t \in [-1,1].$$
   (a) Given an arbitrary $x(t) \in L_2([-1,1])$ what is the closest even function to $\boldsymbol{x}$? That is, solve
$$\min_{\boldsymbol{y}\in\mathcal{E}} \|\boldsymbol{x} - \boldsymbol{y}\|_2^2.$$
A good way to do this is to use the *orthogonality principle* — we know that for the optimal $\hat{\boldsymbol{y}} \in \mathcal{E}$
$$\langle \boldsymbol{x} - \hat{\boldsymbol{y}}, \boldsymbol{z}\rangle = 0 \quad \text{for all} \ \ \boldsymbol{z} \in \mathcal{E}.$$
You might consider filling in the blanks in the following line of reasoning:
$$\begin{aligned}\langle \boldsymbol{x} - \hat{\boldsymbol{y}}, \boldsymbol{z}\rangle &= \int_{-1}^{1} [x(t) - \hat{y}(t)]z(t) \ \mathrm{d}t \\ &= \int_0^1 [x(t) - \hat{y}(t)]z(t) + \cdots \ \mathrm{d}t \\ &= \cdots \\ &= 0 \quad \text{for all } \boldsymbol{z} \in \mathcal{E} \text{ when } \hat{y}(t) = \cdots .\end{aligned}$$

(b) Given an arbitrary $x(t) \in L_2([-1,1])$, solve

$$\min_{\boldsymbol{y} \in \mathcal{O}} \|\boldsymbol{x} - \boldsymbol{y}\|_2^2.$$

(c) Let $\{\phi_k(t),\ k \geq 0\}$ be an orthobasis for $L_2([0,1])$. How can we use this orthobasis on $[0,1]$ to construct an orthobasis $\{\phi_k^e(t),\ k \geq 0\}$ for $\mathcal{E}$? What about an orthobasis $\{\phi_k^o(t),\ k \geq 0\}$ for $\mathcal{O}$? Is $\{\phi_k^e(t),\ k \geq 0\} \cup \{\phi_k^o(t),\ k \geq 0\}$ an orthobasis for all of $L_2([-1,1])$? Why or why not?

4. In this problem, we will develop the computational framework for approximating a continuous-time signal on $[0,1]$ using scaled and shifted version of the classic bell-curve bump:

$$\phi(t) = e^{-t^2}.$$

Fix an integer $N > 0$ and define $\phi_k(t)$ as

$$\phi_k(t) = \phi\left(\frac{t - (k-1/2)/N}{1/N}\right) = \phi\left(Nt - k + 1/2\right)$$

for $k = 1, 2, \ldots, N$. The $\{\phi_k(t)\}$ are a basis for the subspace

$$T_N = \operatorname{span}\left\{\phi_k(t)\right\}_{k=1}^{N}.$$

(a) For a fixed value of N, we can plot all of the $\phi_k(t)$ on the same set of axes in MATLAB using:

```
phi = @(z) exp(-z.^2);
t = linspace(0, 1, 1000);
figure(1); clf
hold on
for kk = 1:N
   plot(t, phi(N*t - kk + 1/2))
end
```

Do this for $N = 10$ and $N = 25$ and turn in your plots.

(b) Since $\{\phi_k(t)\}$ is a basis for T_N, we can write any $y(t) \in T_N$ as

$$y(t) = \sum_{k=1}^{N} a_k \phi_k(t)$$

for some set of coefficients $a_1, \ldots, a_N \in \mathbb{R}^N$. If these coefficients are stacked in an N-vector `a` in MATLAB, we can plot $y(t)$ using

```
t = linspace(0,1,1000);
y = zeros(size(t));
for jj = 1:N
   y = y + a(jj)*phi(N*t - jj + 1/2);
end
plot(t, y)
```

Do this for $N = 4$, and $a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1$ and turn in your plot.

(c) Define the continuous-time signal $x(t)$ on $[0, 1]$ as

$$x(t) = \begin{cases} 4t & 0 \le t < 1/4 \\ -4t + 2 & 1/4 \le t < 1/2 \\ -\sin(20\pi t) & 1/2 \le t \le 1 \end{cases}.$$

Write MATLAB code that finds the closest point $\hat{x}(t)$ in T_N to $x(t)$ for any fixed N. By "closest point", we mean that $\hat{x}(t)$ is the solution to

$$\min_{y \in T_N} \|x(t) - y(t)\|_{L_2([0,1])}.$$

Turn in your code and four plots; one of which has $x(t)$ and $\hat{x}(t)$ plotted on the same set of axes for $N = 5$, and then repeat for $N = 10, 20$, and 50.

Hint: You can create a function pointer for $x(t)$ using

```
x = @(z) (z < 1/4).*(4*z) + (z>=1/4).*(z<1/2).*(-4*z+2) - (z>=1/2).*sin(20*pi*z);
```

and then calculate the continuous-time inner product $\langle x, \phi_k \rangle$ with

```
x_phik = @(z) x(z).*phi(N*z - jj + 1/2);
integral(x_phik, 0, 1)
```

You can use similar code to calculate the entries of the Gram matrix $\langle \phi_j, \phi_k \rangle$. (There is actually a not-that-hard way to calculate the $\langle \phi_j, \phi_k \rangle$ analytically that you can derive if you are feeling industrious — just think about what happens when you convolve a bump with itself.)

5. You want to design an analog filter that has impulse response

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$$

The components you have on hand only allow you implement impulse responses of the form

$$g(t) = \begin{cases} 0, & t < 0, \\ \alpha_1 e^{-t} + \alpha_2 t e^{-t} + \cdots + \alpha_N t^{N-1} e^{-t}, & t \ge 0, \end{cases}$$

where the $\alpha_1, \ldots, \alpha_N$ can be controlled through judicious pole-zero placement. Design optimal filters (i.e. calculate optimal $\{\alpha_k\}$) for $N = 2, 5, 10$. By optimal, we mean

$$\int_0^\infty |h(t) - \hat{h}(t)|^2 \, \mathrm{d}t$$

is minimized. Note: you do not need to remember what I mean by pole-zero placement to solve this problem.

Plot your results on the same axes, along with $h(t)$. Turn in any code you use to solve this problem.