ECE 6250, Fall 2019

Homework #3

Due Wednesday September 11, at the beginning of class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. In class I claimed that for an arbitrary inner product $\langle \cdot, \cdot \rangle$, we can define a valid norm as $\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. Prove this (i.e., show that the properties of an inner product imply that this will satisfy the axioms of a norm). (Hint Cauchy-Schwarz)[1]

3. Below, $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^N$.

   (a) Prove that $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\|_\infty \cdot \|\boldsymbol{y}\|_1$.

   (b) Prove that $\|\boldsymbol{x}\|_1 \leq \sqrt{N} \cdot \|\boldsymbol{x}\|_2$. (Hint: Cauchy-Schwarz)

   (c) Let B_2 be the unit ball for the ℓ_2 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

   $$\max_{\boldsymbol{x} \in B_2} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector x which achieves the maximum. (Hint: Cauchy-Schwarz)

   (d) Let B_∞ be the unit ball for the ℓ_∞ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

   $$\max_{\boldsymbol{x} \in B_\infty} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

[1] You might be concerned that, since the Cauchy-Schwarz inequality already involves the notation $\|\cdot\|$, by using Cauchy-Schwarz we might be assuming the very thing we are trying to prove. However, a careful reading of the proof of Cauchy-Schwarz given in the notes shows that it does not itself assume that $\sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$ define a valid norm. If it makes it less confusing, you can instead think of Cauchy-Schwarz as simply stating that

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}.$$

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(e) Let B_1 be the unit ball for the ℓ_1 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x}\in B_1} \langle \boldsymbol{x}, \boldsymbol{y}\rangle \quad = \quad ????$$

Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

(These last two might require some thought. If you solve them for $N = 2$, it should be easy to generalize.)

4. (a) A square $N \times N$ matrix $\boldsymbol{G}$ is invertible if for every $\boldsymbol{y} \in \mathbb{R}^N$ there is *exactly one* $\boldsymbol{x} \in \mathbb{R}^N$ such that $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{y}$. Show that $\boldsymbol{G}$ is invertible if and only if its columns are linearly independent and $\boldsymbol{G}\boldsymbol{x} \neq \boldsymbol{0}$ for all $\boldsymbol{x} \neq \boldsymbol{0}$.

   (b) Let $\psi_1(t), \ldots, \psi_N(t)$ be continuous-time signals on $t \in \mathbb{R}$, and let $\langle \cdot, \cdot \rangle$ be an arbitrary inner product. Show that the $N \times N$ Grammian

$$\boldsymbol{G} = \begin{bmatrix} \langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_1\rangle & \langle \boldsymbol{\psi}_2, \boldsymbol{\psi}_1\rangle & \cdots & \langle \boldsymbol{\psi}_N, \boldsymbol{\psi}_1\rangle \\ \langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2\rangle & \langle \boldsymbol{\psi}_2, \boldsymbol{\psi}_2\rangle & & \langle \boldsymbol{\psi}_N, \boldsymbol{\psi}_2\rangle \\ \vdots & & \ddots & \vdots \\ \langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_N\rangle & \cdots & & \langle \boldsymbol{\psi}_N, \boldsymbol{\psi}_N\rangle \end{bmatrix},$$

   is invertible if and only if the $\{\boldsymbol{\psi}_n\}$ are linearly independent.