**ECE 6250, Fall 2019**

Homework #1

Due Wednesday, August 28, at the beginning of class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Suppose that a continuous-time signal $x_c(t)$ has the Fourier transform shown below:
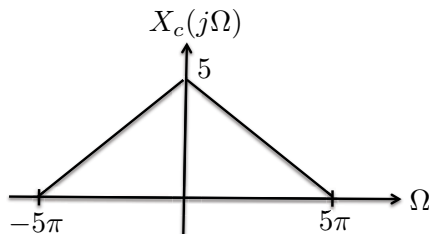
The signal is used as the input to two different systems, one without an anti-aliasing filter:
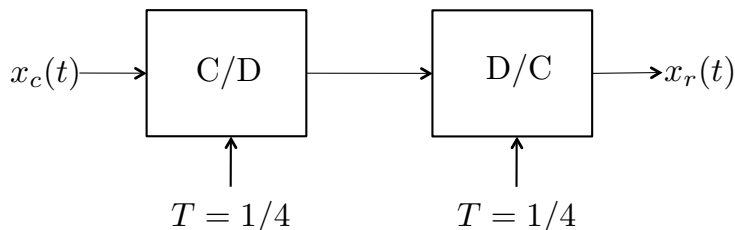
and one with an anti-aliasing filter:
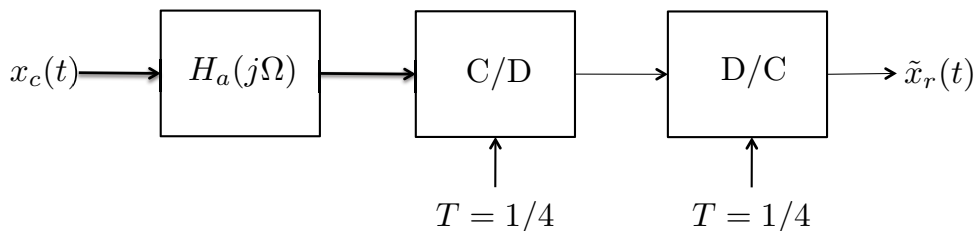
Above, C/D is an ideal continuous-to-discrete converter, D/C is an ideal discrete-to-continuous converter, and

$$H_a(j\Omega) = \begin{cases} 1 & |\Omega| \leq 4\pi \\ 0 & \text{else} \end{cases}.$$

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(a) Calculate the energy of the reconstruction errors in both cases:

$$\|x_c(t) - x_r(t)\|_2^2 \quad \text{and} \quad \|x_c(t) - \tilde{x}_r(t)\|_2^2,$$

where

$$\|f(t)\|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 \ dt.$$

(b) Argue (**rigorously**) that there is no other signal that could come out of the D/C converter that is closer to $x_c(t)$ than $\tilde{x}_r(t)$. (By "closer", we mean that the energy in the error is smaller.)

Knowledge of Parseval's Theorem will definitely help for both parts (a) and (b): for continuous time signnal $x(t)$ with CTFT $X(j\Omega)$,

$$\int_{-\infty}^{\infty} |x(t)|^2 \ dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \ d\Omega.$$

3. Let $x[n]$ be a discrete-time signal whose DTFT $X(e^{j\omega}) = 0$ for $\pi/3 \le |\omega| \le \pi$ which we wish to convert to analog using a standard D/C converter. Suppose that right before we feed this signal into the converter, an adversary changes exactly one sample of $x[n]$ is an unknown way at an unknown location n_0, forming

$$\tilde{x}[n] = \begin{cases} x[n] & n \neq n_0 \\ \text{something else} & n = n_0 \end{cases}.$$

(a) Given our knowledge that the original DTFT $X(e^{j\omega})$ is zero for $\pi/3 \le |\omega| \le \pi$, describe how such an error might be detected by examining the output $y(t)$.

(b) Describe (and sketch a diagram of) a method to correct the error. Hint: one way to do this is with C/D converters which take equally spaced samples but at different *offsets*. That is, suppose the C/D system takes samples with spacing T but offset τ, so $x[n] = x_c(\tau + nT)$.
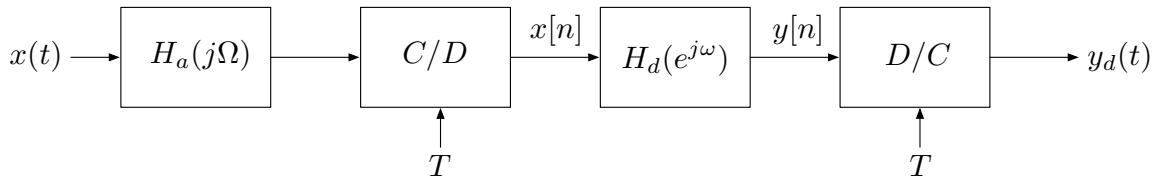
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4. Consider the analog system below

$$x(t) \longrightarrow \boxed{H_c(j\Omega)} \longrightarrow y_c(t)$$
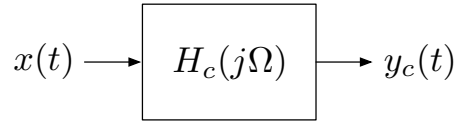
where

$$H_c(j\Omega) = \frac{1}{1000\pi + j\Omega}.$$

We wish to implement a (near) equivalent system in digital using the following architecture:

$$x(t) \longrightarrow \boxed{H_a(j\Omega)} \longrightarrow \boxed{C/D} \xrightarrow{x[n]} \boxed{H_d(e^{j\omega})} \xrightarrow{y[n]} \boxed{D/C} \longrightarrow y_d(t)$$

with T inputs to both C/D and D/C.

where $T = 1/3000$ and $H_a(j\Omega)$ is an antialiasing filter; $H_a(j\Omega) = 1$, $|\Omega| \leq 3000\pi$ and is 0 otherwise. Find $H_d(e^{j\omega})$ such that $y_d(t)$ matches $y_c(t)$ as closely as possible. Sketch your answer.

5. Let $x_c(t)$ be a continuous-time signal which is bandlimited to π/T, and let $x[n]$ correspond to samples taken at the Nyquist rate:

$$x[n] = x_c(nT).$$

As we know, $x_c(t)$ can be perfectly reconstructed from $x[n]$ using sinc interpolation,

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}.$$

This reconstruction process can be modeled as converting $x[n]$ into a spike train, and then passing it through an ideal lowpass filter.
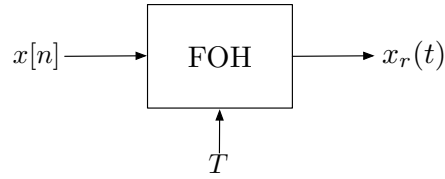
Suppose instead we generate a continuous-time signal using a different interpolation kernel:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\phi(t - nT),$$

where

$$\phi(t) = \begin{cases} 1 - \frac{|t|}{T} & -T \leq t \leq T \\ 0 & |t| > T \end{cases}.$$

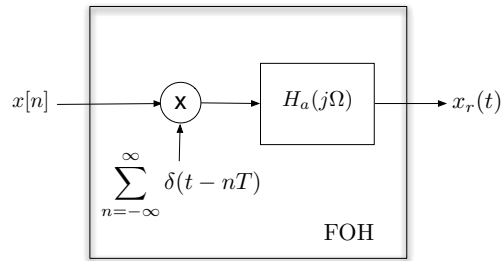We will use FOH to denote the system that maps $x[n]$ to $x_r(t)$:

(a) Sketch $\phi(t)$. Sketch $x_r(t)$ for

$$x[n] = \delta[n+1] + 3\delta[n] - \delta[n-1] + \delta[n-2] + 2\delta[n-3],$$

where $\delta[n]$ is the Kronecker delta function ($\delta[n] = 1$ for $n = 0$ and is zero for all other values of n). How would you describe the action of FOH?

(b) We would like to model FOH as conversion to a spike train followed by an analog filter:
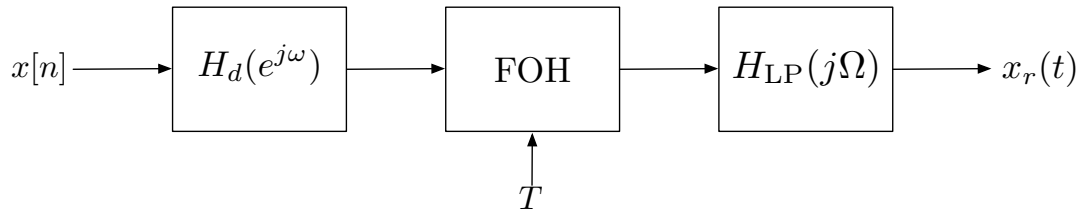

What must the frequency response $H_a(j\Omega)$ be?

(c) Now we precondition the samples $x[n]$ by passing them through a digital filter, and postcondition the output of FOH by passing it through an analog low-pass filter (see the figure below):

$$H_{\mathrm{LP}} = \begin{cases} 1 & |\Omega| \le \pi/T \\ 0 & \text{otherwise} \end{cases}$$

Find $H_d(e^{j\omega})$ so that $x_r(t) = x_c(t)$.

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6. In class we reviewed the fact a sinc in the continuous-time domain transforms into a block in the Fourier domain:

$$h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T} \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad H_T(j\Omega) = \begin{cases} T, & |\Omega| \leq \frac{\pi}{T}, \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}.$$

In this problem, we derive the analogous expressions for the other three types of transforms and use MATLAB to plot the results.

(a) Suppose that a discrete-time signal $h[n]$ has discrete-time Fourier transform (DTFT)

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = \begin{cases} M & |\omega| \leq \pi/M \\ 0 & \pi/M < |\omega| \leq \pi \end{cases}.$$

Find $h[n]$ using the inversion formula

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

Use the `stem` command in MATLAB to plot it for $M = 5$.

(b) Suppose that a continuous-time signal $x(t)$ on the interval $[-1/2, 1/2]$ has Fourier series

$$\alpha_k = \int_{-1/2}^{1/2} x(t) e^{-j2\pi kt}\, dt = \begin{cases} \frac{1}{2B+1} & k = -B, \ldots, 0, \ldots, B \\ 0 & \text{otherwise} \end{cases}$$

Find $x(t)$ using the synthesis formula

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k\, e^{j2\pi kt}.$$

Use the `plot` command in MATLAB to plot it for $B = 7$.

(c) Suppose that a vector $x \in \mathbb{R}^N$ has discrete Fourier transform (DFT)

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N} = \begin{cases} \frac{N}{2B+1} & k = 0, \ldots, B \\ 0 & k = B+1, \ldots, N-B-1 \\ \frac{N}{2B+1} & k = N-B, \ldots, N-1 \end{cases}.$$

Find $x[n]$ using the synthesis formula

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}[k]\, e^{j2\pi kn/N}.$$

Use the `stem` command in MATLAB to plot it for $N = 101$ and $B = 15$.

(Of course the answers to these questions are widely available, but you should actually work it out.)

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7. Suppose that `x` is a column vector in MATLAB with 101 components. Consider the following segment of code:

```
fx = fft(x);
fxi = 10*[fx(1:51); zeros(909,1); fx(52:101)];
xi = ifft(fxi);
```

The 1010 vector `xi` in an interpolation of `x`. To see this, you can set

```
x = randn(101,1);
```

run the code above, and then plot the results:

```
plot(xi)
hold on
stem(1:10:1010,x,'r')
```

Find an explicit expression for the interpolation formula. This should take the form of

$$x_i[n] = \sum_{k=1}^{101} x[k] \text{ (something that depends on } n \text{ and } k).$$