Optimization for Information Systems

Much of modern engineering can be understood as the principled design of systems built for the purpose of manipulating *information*. There are countless examples of such systems – here are a few that we will encounter in this course:

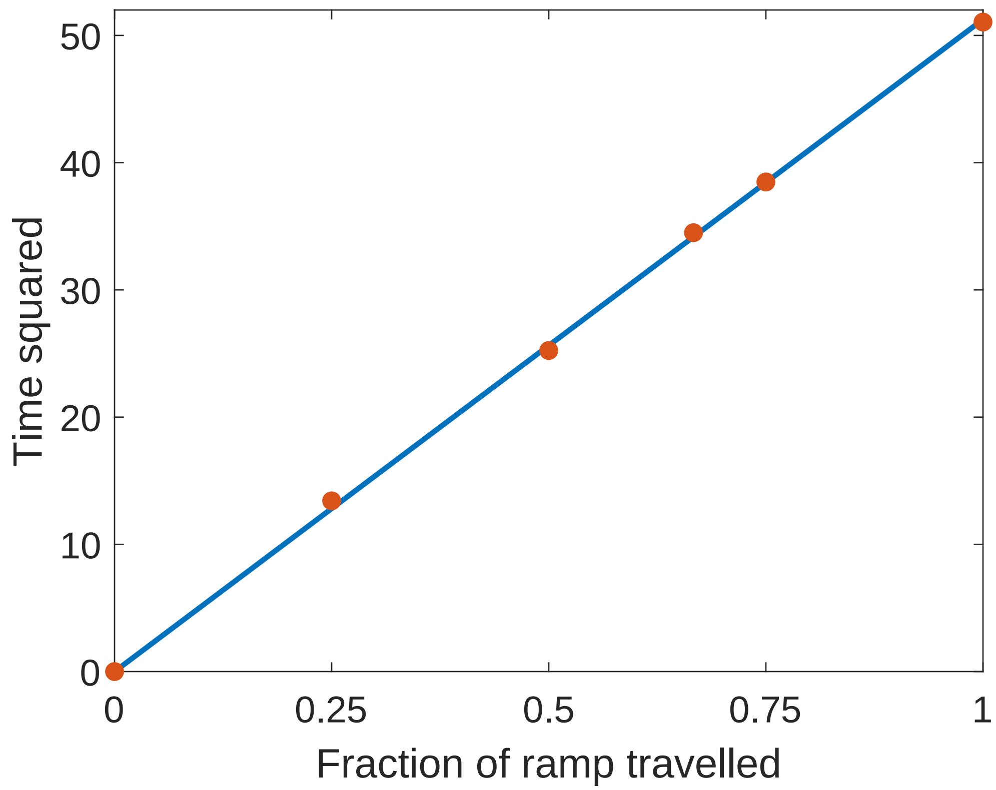
- filters that can remove "noise" to extract a "clean" estimate of a signal;
- systems that process measurements to estimate an object's location (e.g., GPS, RADAR);
- devices for decoding an encoded digital signal that has been corrupted by errors;
- systems that can automatically monitor and control the behavior of another system or device;
- methods for learning to make predictions and/or decisions based on "training data";
- autonomous systems that monitor an environment to perform tasks independently.

Whenever we are designing such a system, we naturally want to do the *best* that we possibly can. We want a filter that minimizes distortion, the optimal estimate of a position, an estimate of a transmission that minimizes the probability of error, etc. Solving such problems involves the use of **optimization**. In this course we will explore the mathematical foundations of this field. We will soon be more formal about what exactly an "optimization problem" consists of, but a useful informal definition is something along the lines of "a problem where we wish to select the best element from some set of possibilities."

Let's give a concrete example of this that demonstrates a possible use of optimization in a real (if not particularly modern) problem. A classic result in physics due to Galileo Galilei is that an object in free-fall experiences uniform acceleration in time. Specifically, the notion of uniform acceleration means that the change in speed should be linearly proportional to the amount of time that has passed, and as a consequence that the distance an object falls should be proportional to the square of the amount of time that has passed. In 1638, Galileo argued in his book *Dialogues Concerning Two New Sciences* that this *ought* to be the case on philosophical grounds (by appealing to Aristotle). But Galileo went further: he argued that one can experimentally verify that bodies in nature really do in fact experience uniform acceleration!

Galileo provided a nice description of how this was done. He constructed an inclined ramp with marks indicating the points $\frac{1}{4}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and all the way down the ramp. He then repeatedly rolled a ball down the ramp, recording the amount of time required for the ball to reach each point (as determined using a water clock that measured the volume of water dripping out of a small spout). Below I illustrate the results from a contemporary recreation of this experiment.[1] We can conclude from visual inspection of the results that the data clearly support Galileo's claim. However, note that there is not a *perfect* agreement between the data and the linear fit to the data that I have also included in the figure. This might raise many questions, including: How did I actually decide on the slope of this linear fit? Is there any sense in which one could determine the "best" fit? While not the approach taken by Galileo, this leads us directly to an example of the role played by optimization in science today.

[1] S. Straulino. "Reconstruction of Galileo Galilei's experiment: The inclined plane." *Physics Education*, 43(3) 316, 2008.



In particular, suppose we observe pairs of points (x_m, y_m) for $m = 1, \ldots, M$, and want to find a function $f(x)$ of the form $f(x) = \alpha x$ such that

$$f(x_m) \approx y_m, \quad m = 1, \ldots, M.$$

To pose this as an optimization problem, we must quantify what we mean by "$\approx$". There are many choices here, but a particulary common one is to measure our "approximation error" using the square of the difference between the observed value y_m and its prediction using $f(x_m)$, averaged over all the observations. Mathematically, we can write this as

$$\frac{1}{M}\sum_{m=1}^{M}(y_m - f(x_m))^2,$$

or in the case where $f(x)$ is of the form $f(x) = \alpha x$,

$$\frac{1}{M} \sum_{m=1}^{M} (y_m - \alpha x_m)^2. \tag{1}$$

We are finally ready to pose this as an optimization problem: if we would like to obtain the "best" linear fit to our data in the sense of squared error, we need to choose α to minimize (1). We can write this as

$$\underset{\alpha \in \mathbb{R}}{\text{minimize}} \ \frac{1}{M} \sum_{m=1}^{M} (y_m - \alpha x_m)^2. \tag{2}$$

This is precisely the problem that I solved to create the optimal linear fit in the figure above. You may remember from calculus how to go about solving this: we are trying to find the minimum of a quadratic function of α, and this will occur precisely at the point where the derivative with respect to α is equal to zero, i.e., when

$$-\frac{1}{M} \sum_{m=1}^{M} 2 x_m (y_m - \alpha x_m) = 0.$$

Simplifying, this yields the following formula for the optimal α, which we will denote by $\widehat{\alpha}$:

$$\widehat{\alpha} = \frac{\sum_{m=1}^{M} x_m y_m}{\sum_{m=1}^{M} x_m^2}.$$

This is an example of a relatively simple optimization problem where we can use tools from calculus to directly find a simple formula for the solution. As we will soon see, however, we are not always so lucky.

Mathematical optimization

We have just encountered our first optimization problem of the course. Optimization problems arise any time we have a collection of elements and wish to select the "best" one (according to some criterion). The process of casting a real world problem as being one of mathematical optimization consists of three main components

1. a set of variables, often called **decision variables**, that we have control over;
2. an **objective function** that maps the decision variables to some quality that we want to maximize (goodness of fit, profit, etc.) or some cost that we want to minimize (error, loss, etc.); and
3. a **constraint set** that dictates restrictions on the decision variables imposed by physical limitations, budgets on resources, design requirements, etc.

In its most general form, we can express such an optimization problem mathematically as

$$\underset{\boldsymbol{x}}{\text{minimize}} \;\; f(\boldsymbol{x}) \;\; \text{subject to} \;\; \boldsymbol{x} \in \mathcal{X}, \tag{3}$$

where $f : \mathcal{X} \to \mathbb{R}$ is our objective function and $\mathcal{X}$ is our constraint set. Compare this with the problem described above in (2).

In order to solve this optimization problem, we must find an $\widehat{\boldsymbol{x}} \in \mathcal{X}$ such that

$$f(\widehat{\boldsymbol{x}}) \leq f(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathcal{X}. \tag{4}$$

We call an $\widehat{\boldsymbol{x}}$ satisfying (4) a **minimizer** of f in $\mathcal{X}$, and a **solution** to the optimization problem (3).

By convention, we will focus only on *minimization* problems, noting that $\widehat{\boldsymbol{x}}$ *maximizes* f in $\mathcal{X}$ if and only if $\widehat{\boldsymbol{x}}$ minimizes $-f$ in $\mathcal{X}$ — thus any maximization problem can be easily turned into an equivalent minimization problem.

There are a number of fundamental questions that arise when considering an optimization problem of the form (3). Our primary interest will be in developing efficient procedures for computing a/the solution to (3). However, we will also need to address more fundamental questions along the way, such as when we can guarantee that a solution even exists, and if so, when we can expect it to be unique. We will begin by exploring these questions in the context of a concrete problem that is ubiquitous in modern science and engineering: least squares optimization.