# 1-Bit Matrix Completion

**Mark A. Davenport**

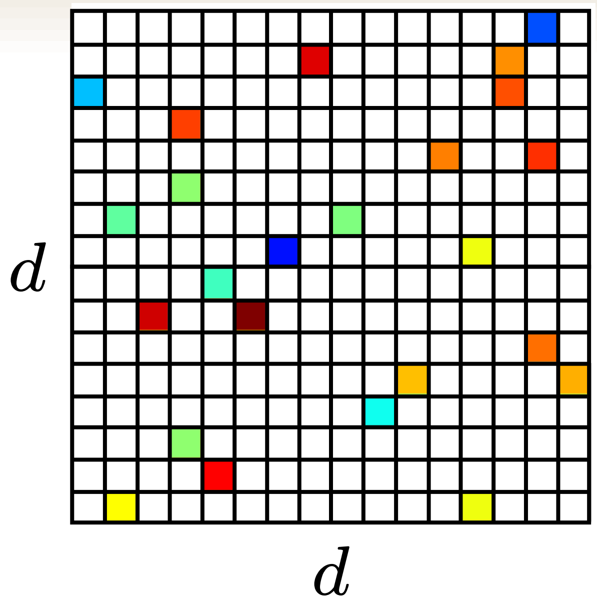
School of Electrical and Computer Engineering
Georgia Institute of Technology

Yaniv Plan

Mary Wootters

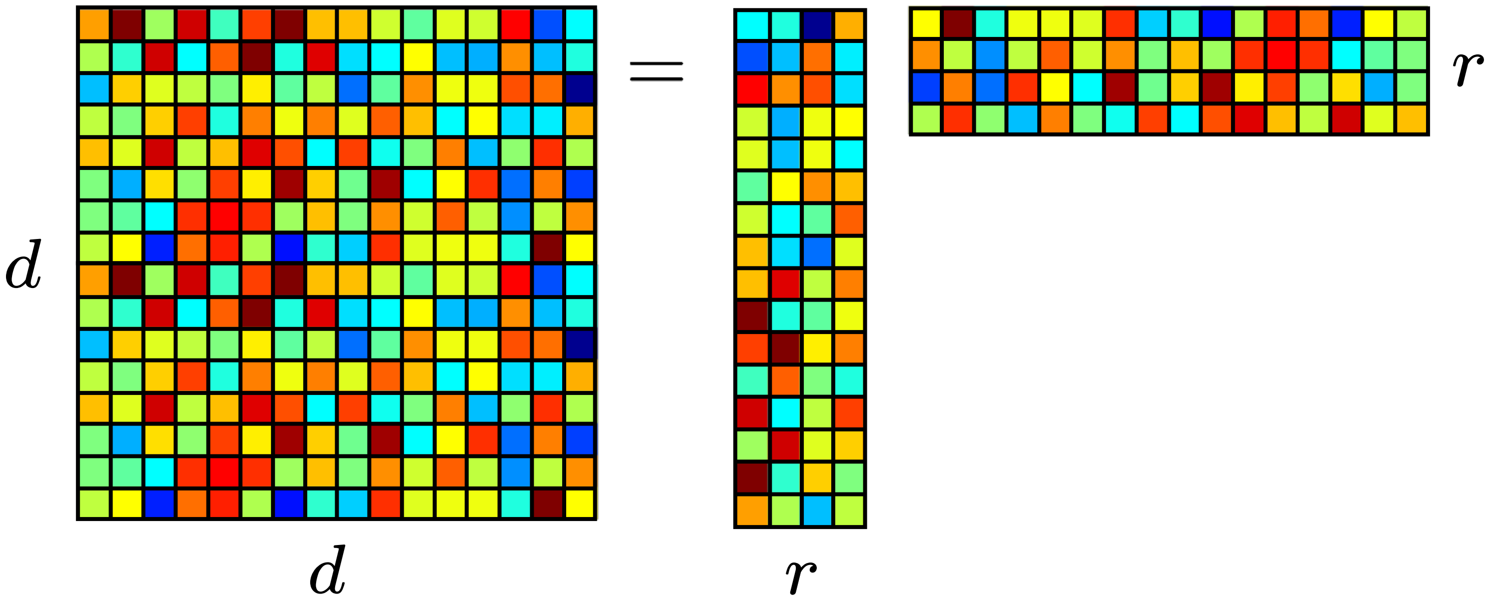
Ewout van den Berg

# Matrix Completion




- When is it possible to recover the original matrix?
- How can we do this efficiently?
- How many samples will we need?

# Low-Rank Matrices




Singular value decomposition:

$$M = U\Sigma V^* \quad \Rightarrow \quad \approx dr \ll d^2$$

degrees of freedom

# Collaborative Filtering

The "Netflix Problem"

$$\boxed{M_{i,j} = \text{how much user } i \text{ likes movie } j}$$

Rank 1 model: $u_i =$ how much user $i$ likes romantic movies

$v_j =$ amount of romance in movie $j$

$$M_{i,j} = u_i v_j$$

Rank 2 model: $w_i =$ how much user $i$ likes zombie movies

$x_j =$ amount of zombies in movie $j$

$$M_{i,j} = u_i v_j + w_i x_j$$

# Beyond Netflix

- Recovery of incomplete survey data
- Analysis of voting data
- Sensor localization
- Quantum state tomography
- …

# Low-Rank Matrix Recovery

Given:

- a $d \times d$ matrix $M$ of rank $r$
- samples of $M$ on the set    : $Y = M$

How can we recover $M$?

$$\widehat{M} = \operatorname*{arg\,inf}_{X : X \quad = Y} \operatorname{rank}(X)$$

Can we replace this with something computationally feasible?

# Nuclear Norm Minimization

*Convex relaxation!*

Replace $\mathrm{rank}(X)$ with $\|X\|_* = \sum_{j=1}^{d} |\sigma_j|$

$$\boxed{\widehat{M} = \mathop{\arg\inf}_{X : X \quad = Y} \|X\|_*}$$

If $| \quad | = O(r\,d\,\log d)$, under certain assumptions, this procedure can recover $M$ !

# Matrix Completion in Practice

- Noise

$$Y = (M + Z)$$

- ***Quantization***
  - Netflix: Ratings are integers between 1 and 5
  - Survey responses: True/False, Yes/No, Agree/Disagree
  - Voting data: Yea/Nay
  - Quantum state tomography: Binary outcomes

Extreme quantization *destroys low-rank structure*

# What's the Problem?

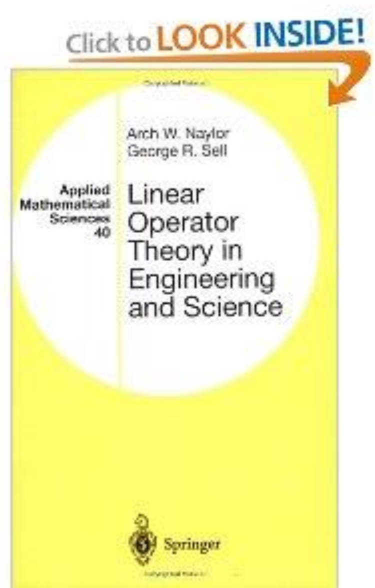
amazon Prime

Mark's Amazon.com | Today's Deals | Gift Cards | Sell | Help

Shop by **Department**

Search Books

**Books** Advanced Search New Releases Best Sellers The New York Times® Best Sellers Children's Books Textbooks Sell You




## Customer Reviews

**Linear Operator Theory in Engineering and Science (Applied Mathematical Sciences)**

**4 Reviews**

| | | |
|---|---|---|
| 5 star: | | (4) |
| 4 star: | | (0) |
| 3 star: | | (0) |
| 2 star: | | (0) |
| 1 star: | | (0) |

**Average Customer Review**
(4 customer reviews)

Share your thoughts with other customers

Create your own review

**i would give 6 stars!**, February 26, 2012

By Char - See all my reviews

**Amazon Verified Purchase** (What's this?)

**This review is from: Linear Operator Theory in Engineering and Science (Applied Mathematical Sciences) (Paperback)**

I'm doing a PhD in econometrics and I need to apply operator theories in constructing a linear or nonlinear operator to help explain individual economic behaviour. This book contains numerous useful ideas and applications with exercises thoroughly designed; one of the questions in the exercise gave me an idea of creating a matrix for describing a nonlinear operator. That question asks for a matrix that describes a second order differential operator and that gave me an idea that taylor series approximation can be used to linearise a nonlinear operator and hence a nonlinear operator may also be described by a matrix.

**Help other customers find the most helpful reviews**

Was this review helpful to you? Yes No

Report abuse | Permalink

Comment

# 1-Bit Matrix Completion

Extreme case

$$Y = \mathrm{sign}(M\ \ )$$

Claim: Recovering $M$ from $Y$ is impossible!

$$M = \begin{bmatrix} \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \end{bmatrix}$$

No matter how many samples we obtain, all we can learn is whether $\lambda > 0$ or $\lambda < 0$

# Is There Any Hope?

If we consider a noisy version of the problem, recovery becomes feasible!

$$Y = \mathrm{sign}(M \quad + Z \quad )$$

$$M + Z = \begin{bmatrix} \lambda + Z_{1,1} & \lambda + Z_{1,2} & \lambda + Z_{1,3} & \lambda + Z_{1,4} \\ \lambda + Z_{2,1} & \lambda + Z_{2,2} & \lambda + Z_{2,3} & \lambda + Z_{2,4} \\ \lambda + Z_{3,1} & \lambda + Z_{3,2} & \lambda + Z_{3,3} & \lambda + Z_{3,4} \\ \lambda + Z_{4,1} & \lambda + Z_{4,2} & \lambda + Z_{4,3} & \lambda + Z_{4,4} \end{bmatrix}$$

Fraction of positive/negative observations tells us something about $\lambda$

Example of the power of *dithering*

# Observation Model

For $(i, j) \in$ we observe

$$Y_{i,j} = \begin{cases} +1 & \text{with probability } f(M_{i,j}) \\ -1 & \text{with probability } 1 - f(M_{i,j}) \end{cases}$$

If $f$ behaves like a CDF, then this is equivalent to

$$Y_{i,j} = \text{sign}(M_{i,j} + Z_{i,j})$$

where $Z_{i,j}$ is drawn according to a suitable distribution

We will assume that is drawn uniformly at random

# Examples

- Logistic regression / Logistic noise

$$f(x) = \frac{e^x}{1+e^x}$$

$Z_{i,j} \sim$ logistic distribution

- Probit regression / Gaussian noise

$$f(x) = \Phi(x/\sigma)$$

$$Z_{i,j} \sim \mathcal{N}(0, \sigma^2)$$

# Maximum Likelihood Estimation

Log-likelihood function:

$$F(X) = \sum_{(i,j)\in \quad +} \log(f(X_{i,j})) + \sum_{(i,j)\in \quad -} \log(1 - f(X_{i,j}))$$

$$\boxed{\begin{aligned} \widehat{M} &= \arg\max_{X} F(X) \\ &\qquad \text{s.t.} \;\; \operatorname{rank}(X) \le r \end{aligned}}$$

# Maximum Likelihood Estimation

Log-likelihood function:

$$F(X) = \sum_{(i,j)\in \;_{+}} \log(f(X_{i,j})) + \sum_{(i,j)\in \;_{-}} \log(1 - f(X_{i,j}))$$

$$\begin{aligned} \widehat{M} &= \arg\max_{X} F(X) \\ &\quad \text{s.t. } \frac{1}{d\alpha}\|X\|_{*} \leq \sqrt{r} \\ &\quad\quad \|X\|_{\infty} \leq \alpha \end{aligned}$$

# Recovery of the Matrix

***Theorem*** *(Upper bound achieved by convex ML estimator)*

Assume that $\frac{1}{d\alpha}\|M\|_* \leq \sqrt{r}$ and $\|M\|_\infty \leq \alpha$. If is chosen at random with $\mathbb{E}|\quad| = m > d\log d$, then with high probability

$$\frac{1}{d^2}\|\widehat{M} - M\|_F^2 \leq C\alpha L_\alpha \beta_\alpha \sqrt{\frac{rd}{m}}$$

where

$$L_\alpha := \sup_{|x|\leq\alpha} \frac{|f'(x)|}{f(x)(1-f(x))} \qquad \beta_\alpha := \sup_{|x|\leq\alpha} \frac{f(x)(1-f(x))}{(f'(x))^2}$$

Is this bound tight?

# Recovery of the Matrix

***Theorem*** *(Upper bound achieved by convex ML estimator)*

Assume that $\frac{1}{d\alpha}\|M\|_* \leq \sqrt{r}$ and $\|M\|_\infty \leq \alpha$. If  is chosen at random with $\mathbb{E}|\ \ | = m > d \log d$, then with high probability

$$\frac{1}{d^2}\|\widehat{M} - M\|_F^2 \leq C\alpha L_\alpha \beta_\alpha \sqrt{\frac{rd}{m}}$$

***Theorem*** *(Lower bound on any estimator)*

There exist $M$ satisfying the assumptions above such that for any set  with $|\ \ | = m$, we have (under mild technical assumptions) that

$$\inf_{\widehat{M}} \mathbb{E}\left[\frac{1}{d^2}\|\widehat{M} - M\|_F^2\right] \geq c\alpha\sqrt{\beta_{\frac{3}{4}\alpha}}\sqrt{\frac{rd}{m}}$$

# Logistic Model

$$L_\alpha = 1 \qquad \beta_\alpha \approx e^\alpha$$

***Theorem*** *(Upper bound achieved by convex ML estimator)*

$$\frac{1}{d^2}\|\widehat{M} - M\|_F^2 \leq C\alpha e^\alpha \sqrt{\frac{rd}{m}}$$

***Theorem*** *(Lower bound on any estimator)*

$$\inf_{\widehat{M}} \mathbb{E}\left[\frac{1}{d^2}\|\widehat{M} - M\|_F^2\right] \geq c\alpha e^{\frac{3}{8}\alpha}\sqrt{\frac{rd}{m}}$$

# Probit Model

$$L_\alpha \approx \frac{\frac{\alpha}{\sigma} + 1}{\sigma} \qquad \beta_\alpha \approx \sigma^2 e^{\alpha^2/2\sigma^2}$$

***Theorem** (Upper bound achieved by convex ML estimator)*

$$\frac{1}{d^2}\|\widehat{M} - M\|_F^2 \leq C\left(\frac{\alpha}{\sigma} + 1\right) e^{\alpha^2/2\sigma^2} \sigma\alpha\sqrt{\frac{rd}{m}}$$

Two regimes

- High signal-to-noise ratio: $\sigma \leq \alpha$
- Low signal-to-noise ratio: $\sigma \geq \alpha$

Compare to how well we can estimate $M$ from unquantized, noisy measurements

# Probit Model (High SNR)

***Theorem*** *(Upper bound achieved by convex ML estimator)*

$$\frac{1}{d^2}\|\widehat{M} - M\|_F^2 \leq C\alpha^2 e^{\alpha^2/2\sigma^2}\sqrt{\frac{rd}{m}}$$

***Theorem*** *(Lower bound on any estimator with unquantized measurements)*

$$\inf_{\widehat{M}} \mathbb{E}\left[\frac{1}{d^2}\|\widehat{M} - M\|_F^2\right] \geq c\alpha\sigma\sqrt{\frac{rd}{m}}$$

# Probit Model (Low SNR)

***Theorem*** *(Upper bound achieved by convex ML estimator)*

$$\frac{1}{d^2}\|\widehat{M} - M\|_F^2 \leq C\alpha\sigma\sqrt{\frac{rd}{m}}$$

***Theorem*** *(Lower bound on any estimator with unquantized measurements)*

$$\inf_{\widehat{M}} \mathbb{E}\left[\frac{1}{d^2}\|\widehat{M} - M\|_F^2\right] \geq c\alpha\sigma\sqrt{\frac{rd}{m}}$$

More noise can lead to *improved* performance!

# Recovery of the Distribution

- It is also possible to establish bounds concerning the recovery of the distribution $f(M)$, i.e., the matrix where each entry gives us the probability of observing +1 when we sample that entry
- We obtain matching upper and lower bounds on the average Hellinger distance between $f(M)$ and $f(\widehat{M})$
- When $\lim_{\alpha \to \infty} L_\alpha < \infty$, we can recover the distribution $f(M)$ *without any assumptions on* $\|M\|_\infty$
  - logistic model
  - *not* probit model
  - any model where the noise has heavy tails

# Tiny Sketch of Proof of Upper Bound

Recall that we maximize the log-likelihood $F(X)$

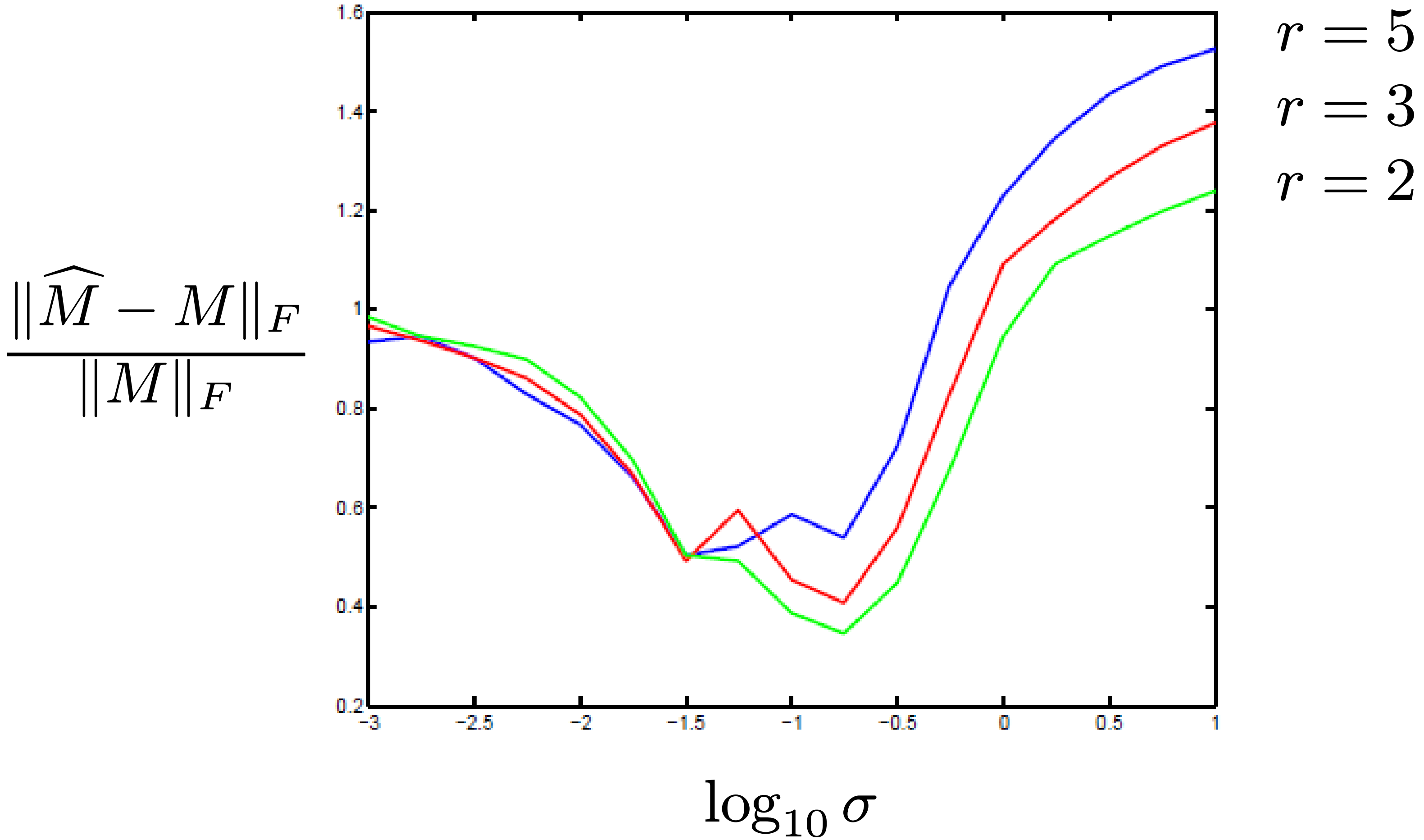
- For a fixed matrix $X$, $\mathbb{E}[F(M) - F(X)] = c \cdot D(f(X) || f(M))$
- Lemma: Let $K = \{X : \frac{1}{d\alpha} \|X\|_* \leq \sqrt{r}\}$. With high probability, $\sup_{X \in K} |F(X) - \mathbb{E}F(X)| \leq \delta$
- By definition, $F(\widehat{M}) \geq F(M)$

$$\begin{aligned} 0 &\geq F(M) - F(\widehat{M}) \\ &\geq \mathbb{E}\left[F(M) - F(\widehat{M})\right] - 2\delta \\ &= c \cdot D(f(\widehat{M}) || f(M)) - 2\delta \end{aligned}$$

- Thus, $D(f(\widehat{M}) || f(M)) \leq \frac{2}{c}\delta$

# Synthetic Simulations

$$d = 500 \qquad m = .15d^2$$



# MovieLens Data Set

- 100,000 movie ratings on a scale from 1 to 5
- Convert to binary outcomes by comparing each rating to the average rating in the data set
- Evaluate by checking if we predict the correct sign
- Training on 95,000 ratings and testing on remainder
  - "standard" matrix completion: 60% accuracy

    1: 64%  2: 56%  3: 44%  4: 65%  5: 74%

  - 1-bit matrix completion: 73% accuracy

    1: 79%  2: 73%  3: 58%  4: 75%  5: 89%

# Conclusions

- 1-bit matrix completion is hard!
- What did you really expect?
- Sometimes 1-bit is all we can get…
- We have algorithms that are near optimal
- Open questions
  - Are there simpler/better/faster/stronger algorithms?
  - What about 2.32-bit matrix completion?

Thank You!