Reconstruction and Cancellation of Sampled Multiband Signals Using Discrete Prolate Spheroidal Sequences

Mark A. Davenport

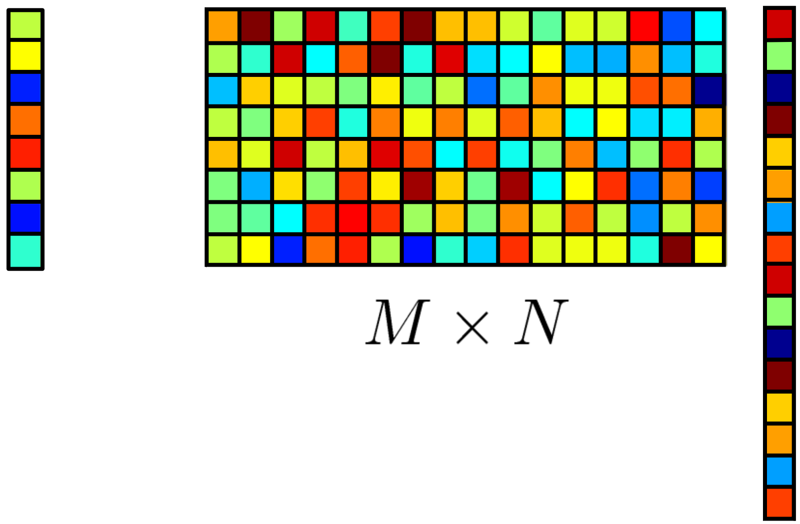
Stanford University
Department of Statistics

Michael B. Wakin

Colorado School of Mines
Division of Engineering

Compressive Sensing (CS)

$$y = \Phi x$$



Can we really acquire analog signals with “CS”?

Potential Obstacles

Obstacle 1: CS is discrete, finite-dimensional

Obstacle 2: Analog sparse representations

Obstacle 1

Obstacle 1: CS is discrete, finite-dimensional

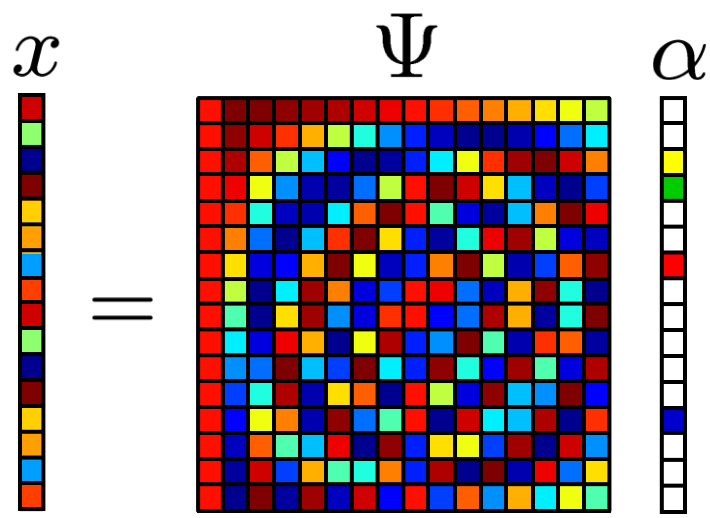
For any bandlimited signal $x(t)$,

$$\begin{aligned} y[m] &= \langle \phi_m(t), x(t) \rangle \\ &= \sum_{n=-\infty}^{\infty} x[n] \, \langle \phi_m(t), \mathrm{sinc}(t/T_s - n) \rangle \end{aligned}$$

For many practical architectures, $y[m]$ will depend on only a finite window of $x[n]$.

Obstacle 2

Obstacle 2: Analog sparse representations

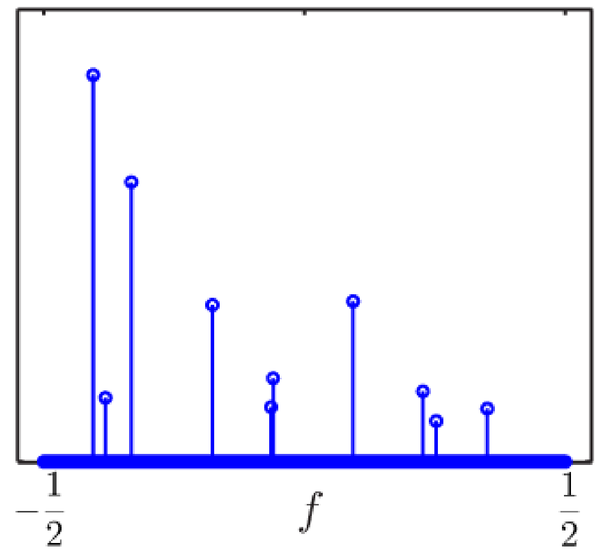


The structure of Ψ will derive from a continuous-time signal model.

Candidate Analog Signal Models

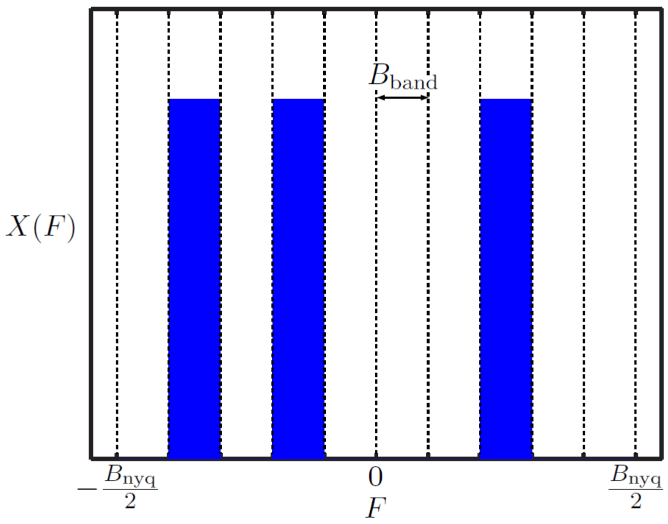
Multitone model:

- periodic signal
- DFT with S *tones*
- unknown *amplitude*



Multiband model:

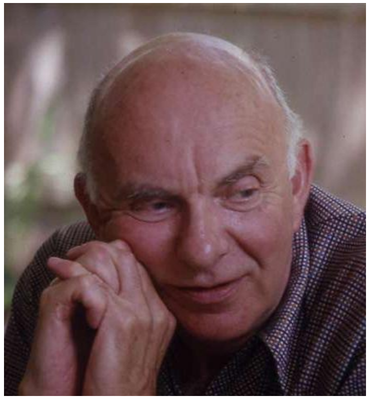
- aperiodic signal
- DTFT with K *bands* of bandwidth B_{band}
- unknown *spectra*



Discrete Prolate Spheroidal Sequences (DPSS's)

DPSS's (Slepian sequences)

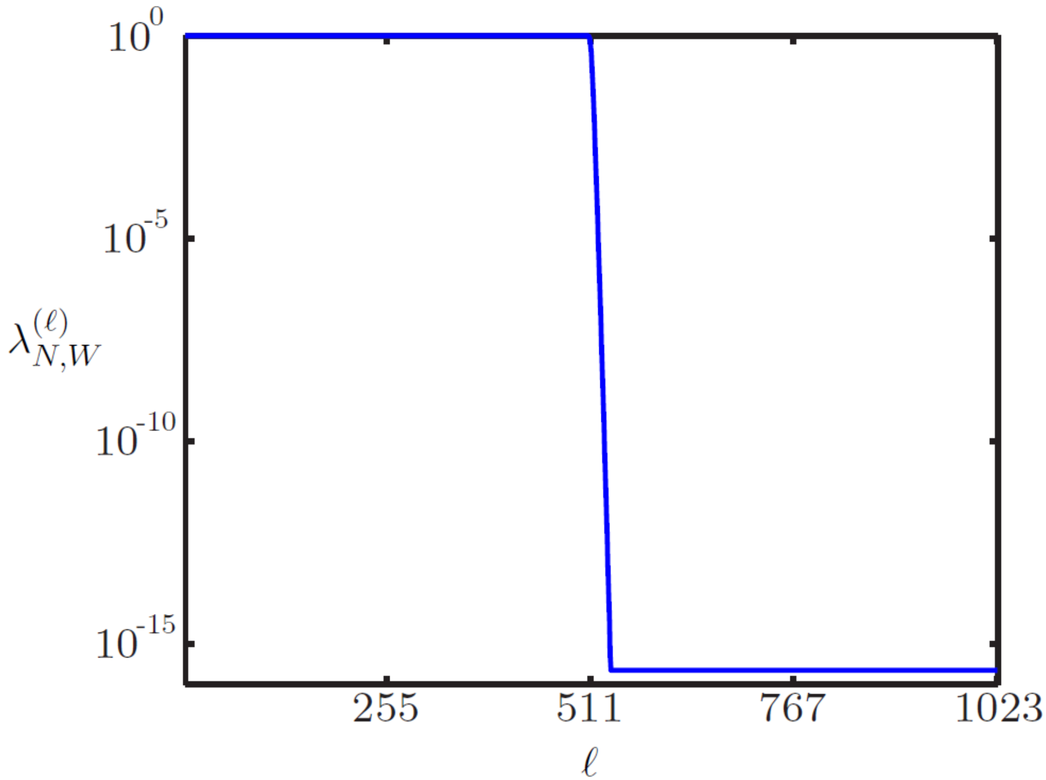
Given N and $W \le \frac{1}{2}$, the DPSS's are a collection of N real-valued discrete-time sequences $s_{N,W}^{(0)}, s_{N,W}^{(1)}, \ldots, s_{N,W}^{(N-1)}$ such that for all ℓ



$$\mathcal{B}_W(\mathcal{T}_N(s_{N,W}^{(\ell)})) = \lambda_{N,W}^{(\ell)} s_{N,W}^{(\ell)}.$$

The DPSS's are perfectly bandlimited, but when $\lambda_{N,W}^{(\ell)} \approx 1$ they are highly concentrated in time.

DPSS Eigenvalue Concentration

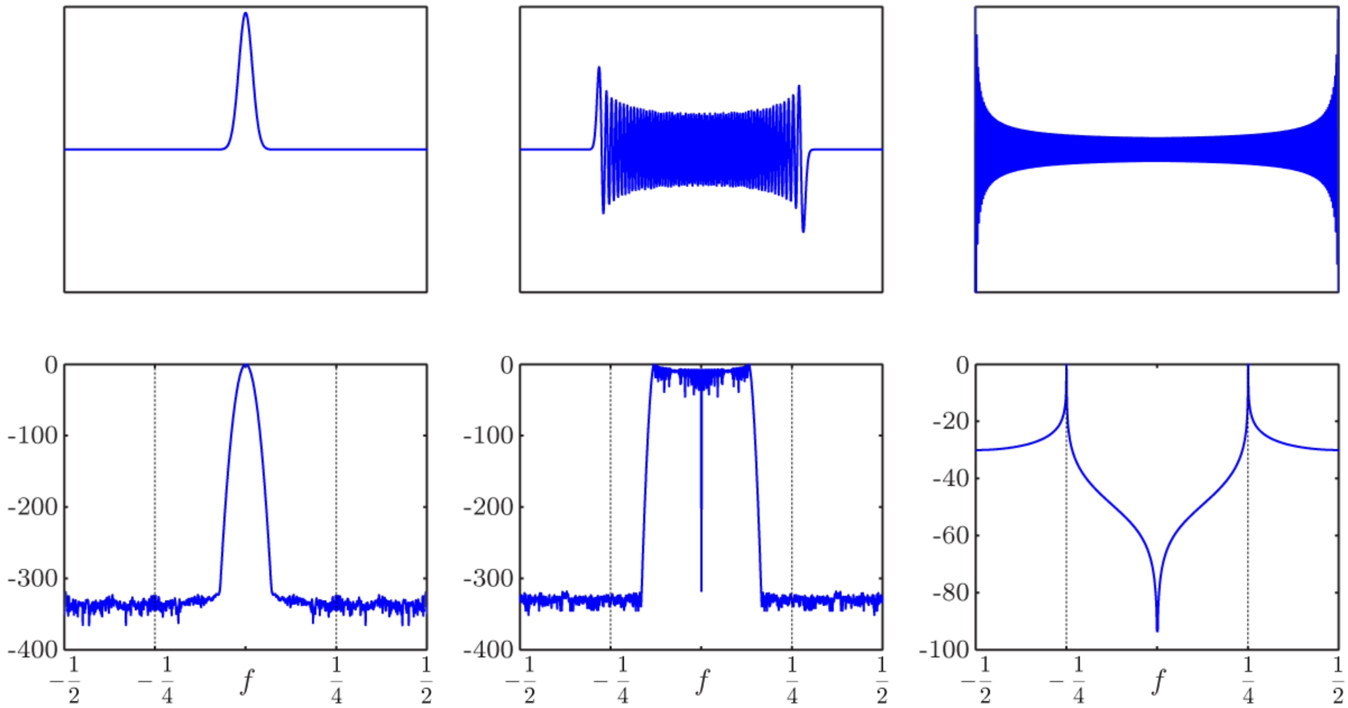


$$N = 1024$$
$$W = \frac{1}{4}$$

The first $\approx 2NW$ eigenvalues ≈ 1.
The remaining eigenvalues ≈ 0.

DPSS Examples

$$N = 1024 \qquad W = \frac{1}{4}$$



$\ell = 0$ $\qquad$ $\ell = 127$ $\qquad$ $\ell = 511$

Why DPSS's?

Suppose that we wish to minimize

$$\frac{1}{2W} \cdot \int_{-W}^{W} \|e_f - P_Q e_f\|_2^2 \, df$$

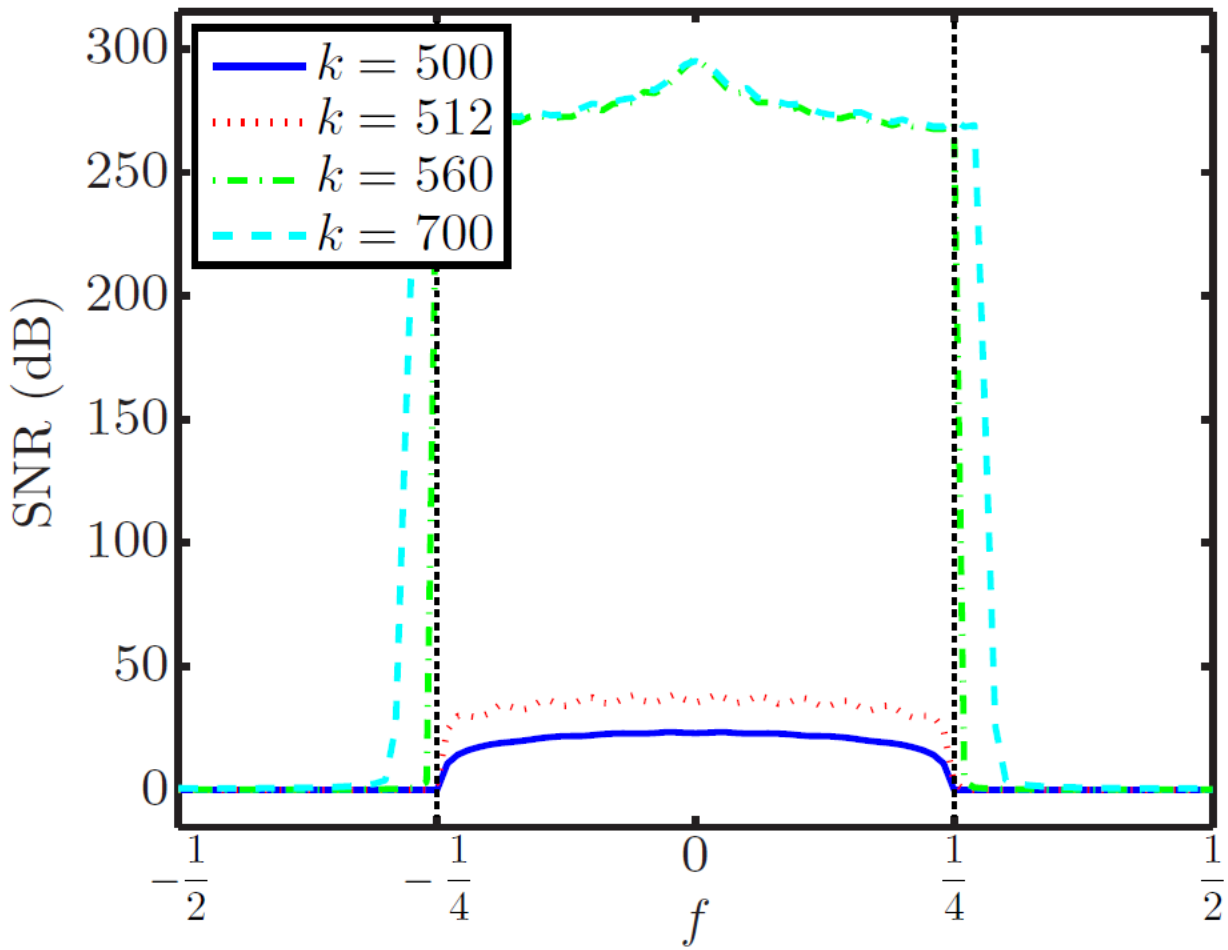
over Q where $e_f := \left[e^{j2\pi f 0}, e^{j2\pi f}, \ldots, e^{j2\pi f (N-1)}\right]^T$.

Optimal subspace of dimension k is the one spanned by the first k DPSS vectors.

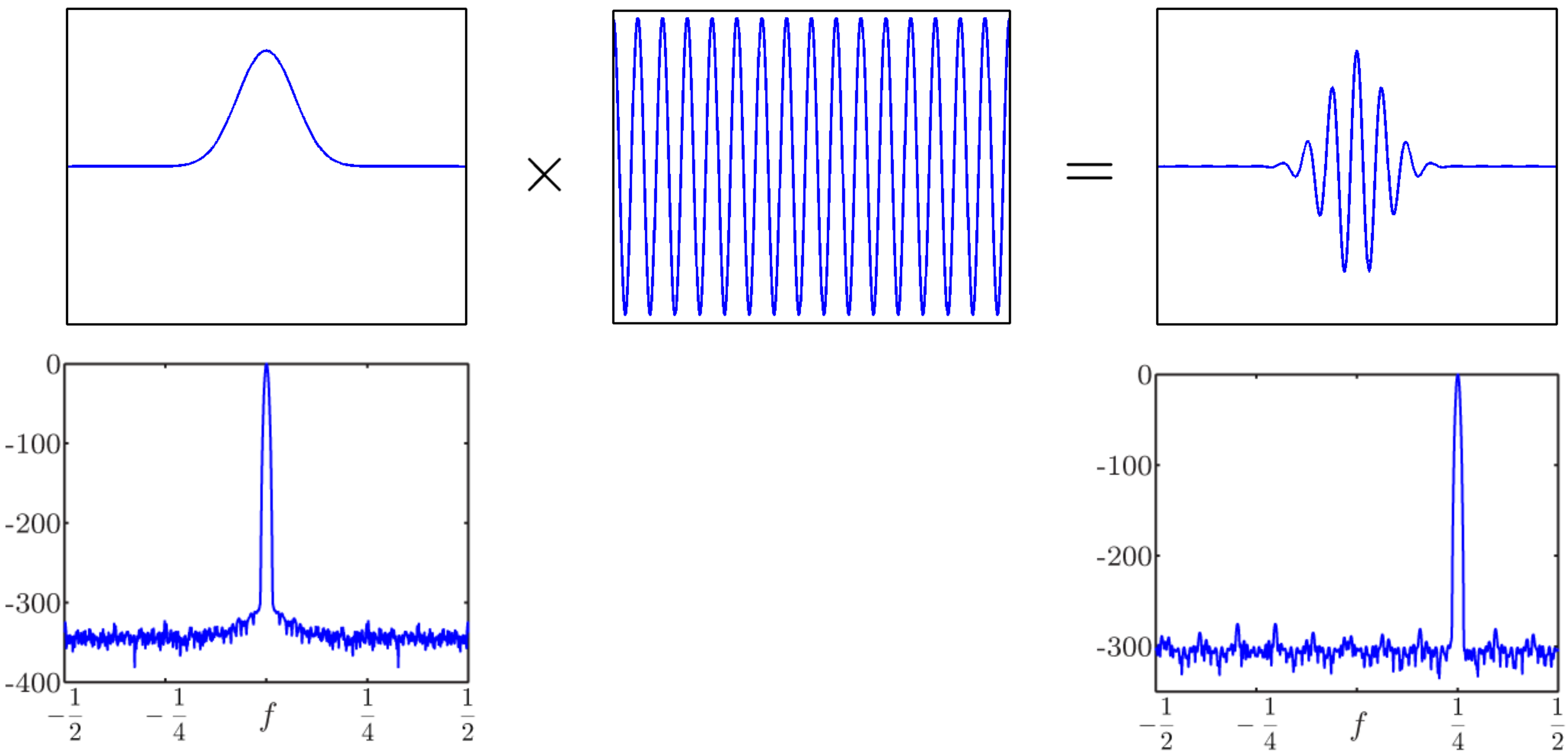
$$\frac{1}{2W} \cdot \int_{-W}^{W} \|e_f - P_Q e_f\|_2^2 \, df = \frac{1}{2W} \sum_{\ell=k}^{N-1} \lambda_{N,W}^{(\ell)}$$

Approximation Performance

$$\mathrm{SNR} = 20\log_{10}\left(\frac{\|e_f\|}{\|e_f - P_Q e_f\|}\right) \mathrm{dB}$$



DPSS's for Passband Signals



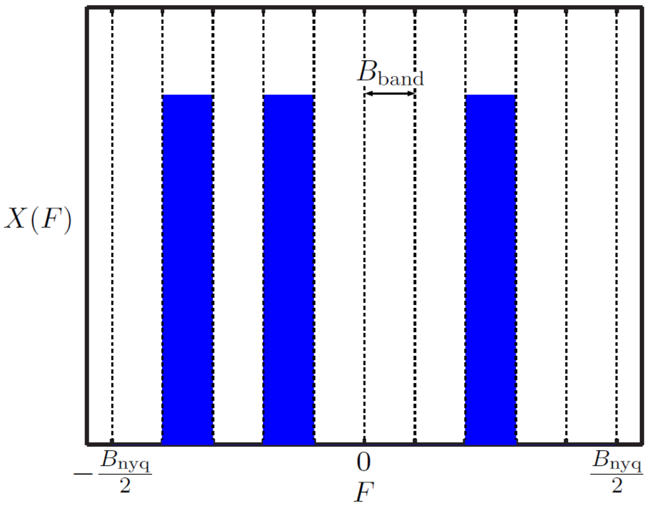
DPSS Dictionaries for CS

Construct dictionary Ψ as

$$\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_J]$$

where Ψ_i is the matrix of the first k DPSS's modulated to $f_i = -\frac{1}{2} + \left(i + \frac{1}{2}\right)(B_{\text{band}}/B_{\text{nyq}})$.

Ψ sparsely and accurately represents *most* sampled multiband signals.



DPSS Dictionaries and the RIP

Let $W = \frac{1}{2}(B_{\mathrm{band}}/B_{\mathrm{nyq}})$. Suppose that Φ is sub-Gaussian and that the Ψ_i are constructed with $k = (1-\epsilon)2NW$. If

$$M \geq CS \log(N/S)$$

then with high probability $\Phi\Psi$ will satisfy

$$(1-\delta)\|\alpha\|_2^2 \leq \|\Phi\Psi\alpha\|_2^2 \leq (1+\delta)\|\alpha\|_2^2$$

for all S-sparse α.

K occupied bands $\Rightarrow$ $S \approx KNB_{\mathrm{band}}/B_{\mathrm{nyq}}$

$$\boxed{\frac{M}{N} \geq C' \frac{KB_{\mathrm{band}}}{B_{\mathrm{nyq}}} \log\left(\frac{B_{\mathrm{nyq}}}{KB_{\mathrm{band}}}\right)}$$

Block-Sparse Recovery

Nonzero coefficients of α should be clustered in blocks according to the occupied frequency bands

$$x = [\Psi_1, \Psi_2, \ldots, \Psi_J] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_J \end{bmatrix}$$

This can be leveraged to reduce the required number of measurements and improve performance through "model-based CS"

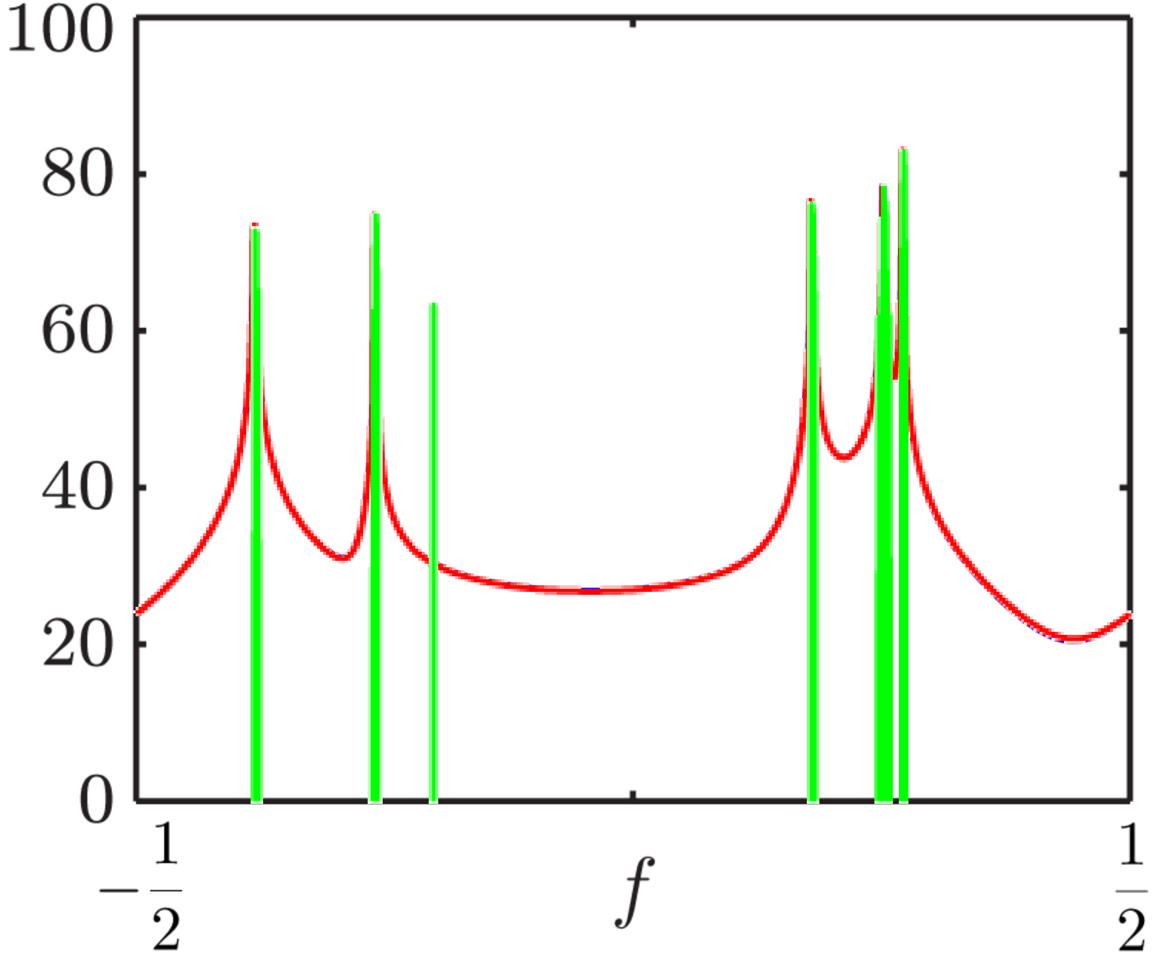
– Baraniuk et al. [2008, 2009, 2010]

– Blumensath and Davies [2009, 2011]

Recovery: DPSS vs DFT

$$N = 1024 \quad M = 128 \quad K = 5 \quad \frac{B_{\text{band}}}{B_{\text{nyq}}} = \frac{1}{512}$$

$$S \approx 45$$

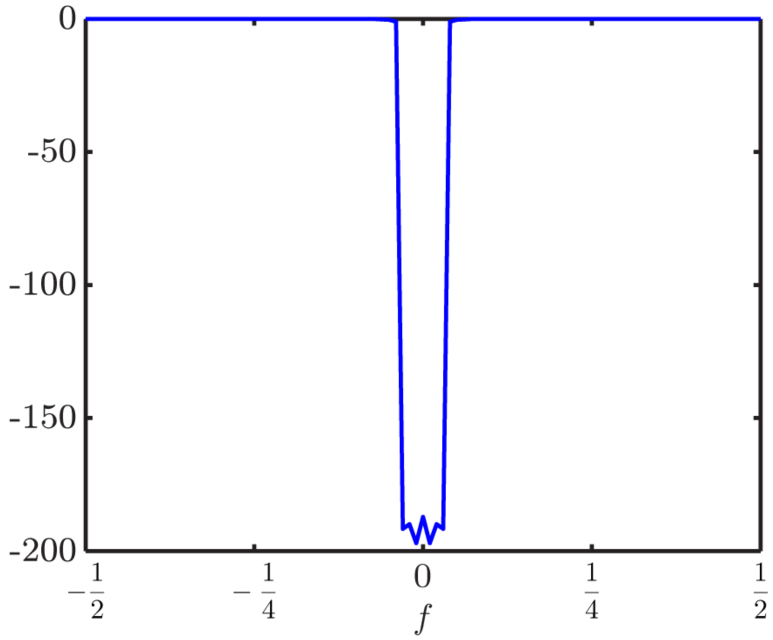


$$\text{DPSS}: \quad \text{SNR} = 54\text{dB}$$

$$\text{DFT}: \quad \text{SNR} = 12\text{dB}$$

Interference Cancellation

DPSS's can be used to cancel bandlimited interferers *without reconstruction.*



$$P = I - \Phi\Psi_i(\Phi\Psi_i)^\dagger$$

Extremely useful in *compressive signal processing* applications.

Summary

- DPSS's can be used to efficiently represent *most* sampled multiband signals
  - knowledge of occupied bands not necessary a priori
  - far superior to DFT
- Two types of error: *approximation* + *reconstruction*
  - approximation: small for most signals
  - reconstruction: zero for DPSS-sparse vectors
  - delicate balance in practice, but there is a sweet spot
- Applications
  - signal reconstruction
  - interference cancellation
  - compressive signal processing