Adaptive envelope estimation of sparse signals

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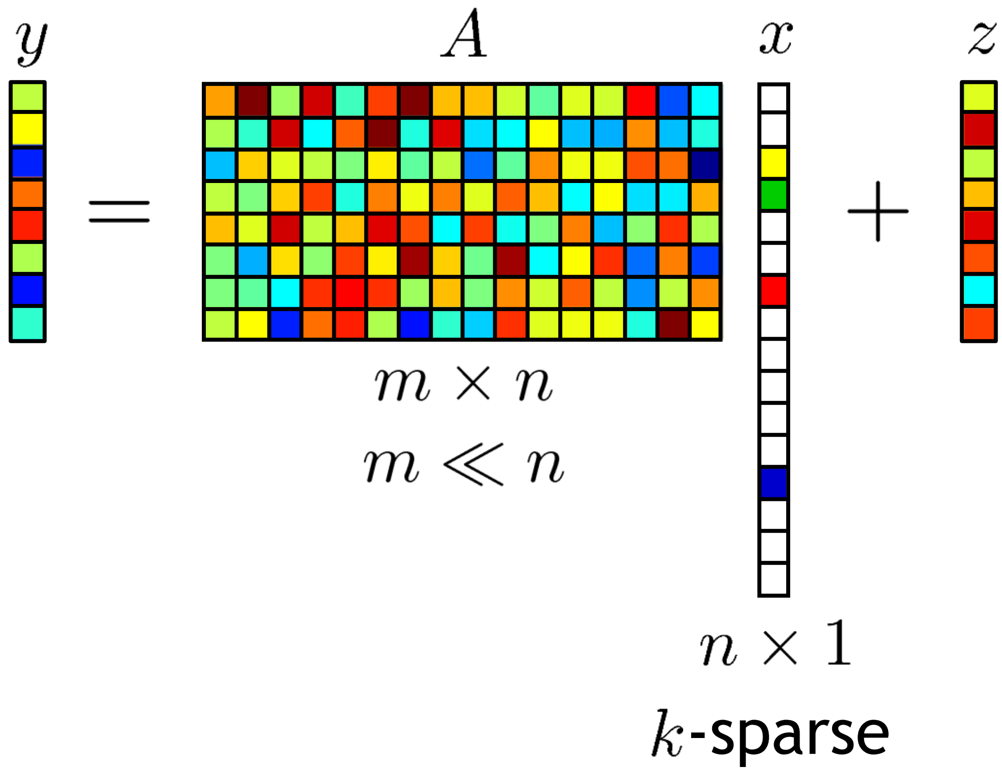
Compressive Sensing



When (and how well) can we estimate x from the measurements y?

Room For Improvement?

$$y_i = \langle a_i, x \rangle + z_i$$

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- We are using most of our "sensing power" to sense entries that aren't even there!
- Tremendous loss in signal-to-noise ratio (SNR)
- It's hard to imagine any way to avoid this…

How Well Can We Estimate x ?

$$y = Ax + z \qquad\qquad z \sim \mathcal{N}(0, \sigma^2 I)$$

Suppose that A has unit-norm rows.

There exist matrices A such that for any x with $\|x\|_0 \leq k$

$$\mathbb{E}\,\|\widehat{x} - x\|_2^2 \leq C\,\frac{n}{m}k\sigma^2 \log n.$$

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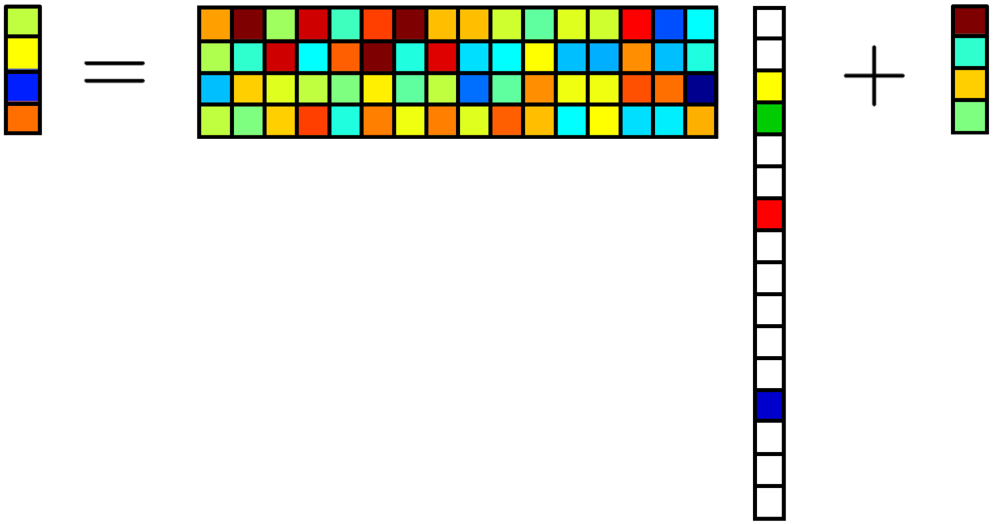
For *any* choice of A and *any* possible recovery algorithm, there exists an x with $\|x\|_0 \leq k$ such that

$$\mathbb{E}\,\|\widehat{x} - x\|_2^2 \geq C'\,\frac{n}{m} k\sigma^2 \log(n/k).$$

Adaptive Sensing

Think of sensing as a game of 20 questions



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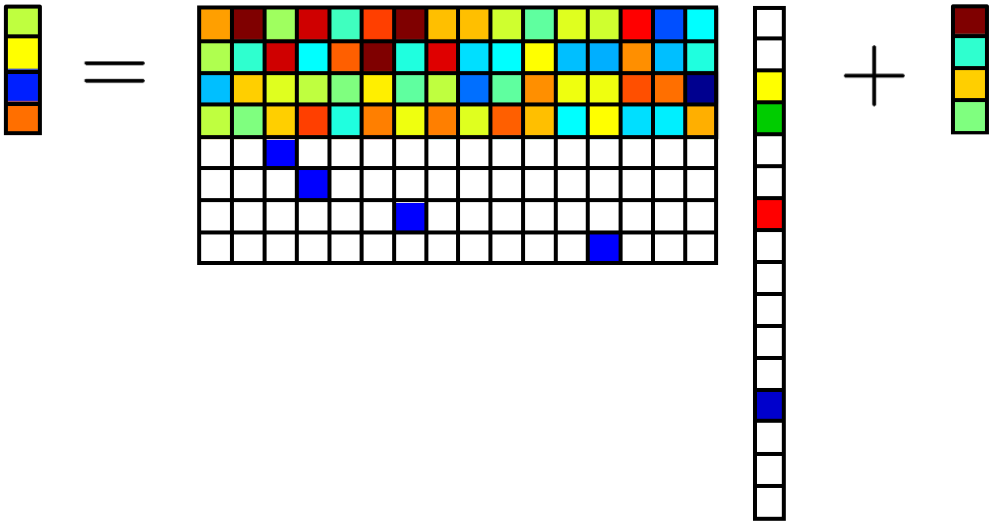

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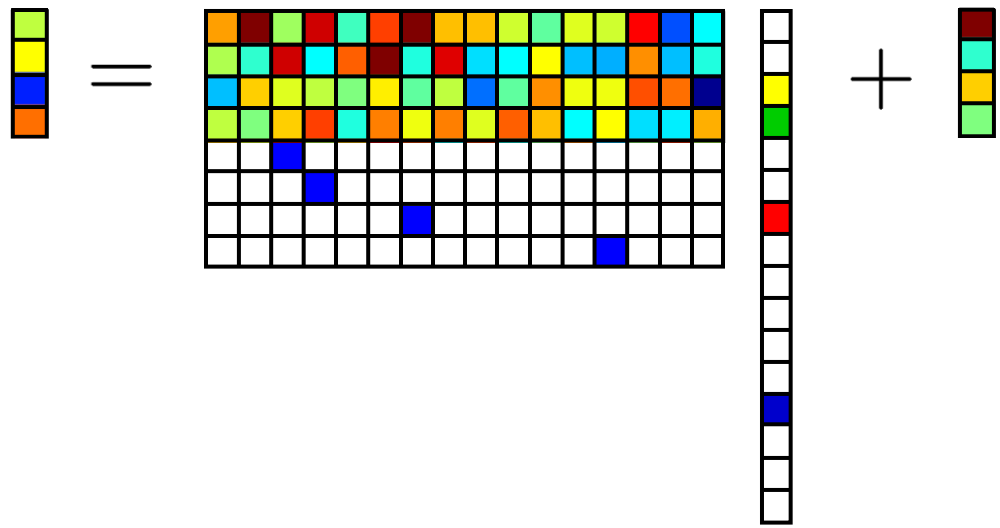
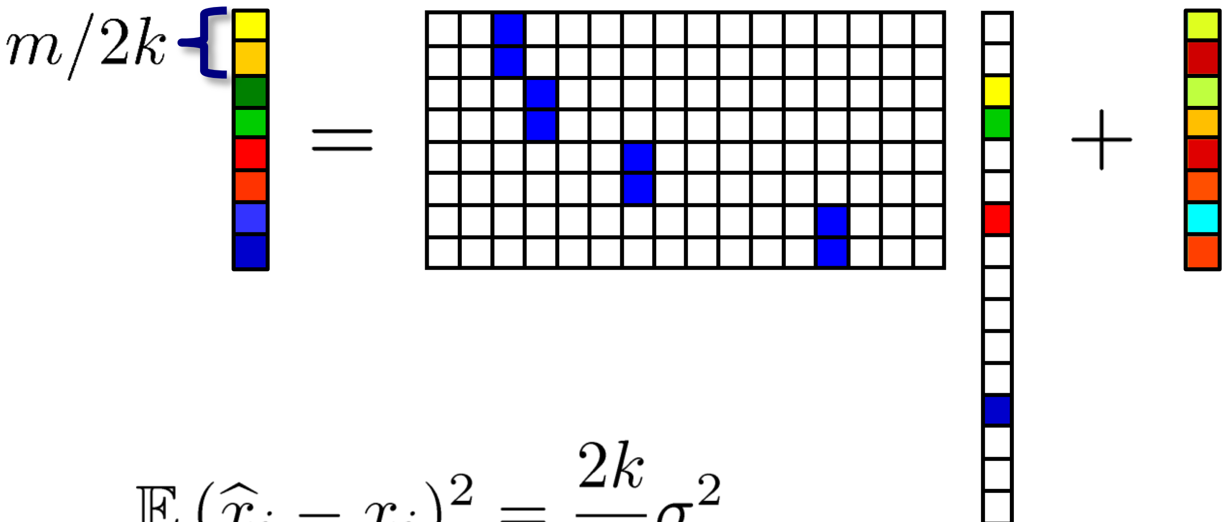


Simple strategy: Use $m/2$ measurements to find the support (envelope), and the remainder to estimate the values.

Thought Experiment

Suppose that after $m/2$ measurements we have perfectly estimated the locations of the nonzeros.



$$\mathbb{E}\left(\widehat{x}_i - x_i\right)^2 = \frac{2k}{m}\sigma^2$$

$$\mathbb{E}\,\|\widehat{x} - x\|_2^2 = \frac{2k}{m}k\sigma^2 \ll \frac{n}{m}k\sigma^2 \log n$$

Limits of Adaptivity

Suppose we have a budget of m measurements of the form $y_i = \langle a_i, x \rangle + z_i$ where $\|a_i\|_2 = 1$ and $z_i \sim \mathcal{N}(0, \sigma^2)$

The vector a_i can have an arbitrary dependence on the measurement history, i.e., $(a_1, y_1), \ldots, (a_{i-1}, y_{i-1})$

Theorem
There exist x with $\|x\|_0 \leq k$ such that for *any* adaptive measurement strategy and *any* recovery procedure $\widehat{x}$,

$$\mathbb{E}\,\|\widehat{x}(y) - x\|_2^2 \geq \frac{4}{7}\frac{n}{m}k\sigma^2.$$

Thus, adaptivity seemingly does *not* significantly help!

[Arias-Castro, Candès, and Davenport - 2011]

Adaptivity In Practice

Suppose that $k = 1$ and that $x_{j^*} = \mu$

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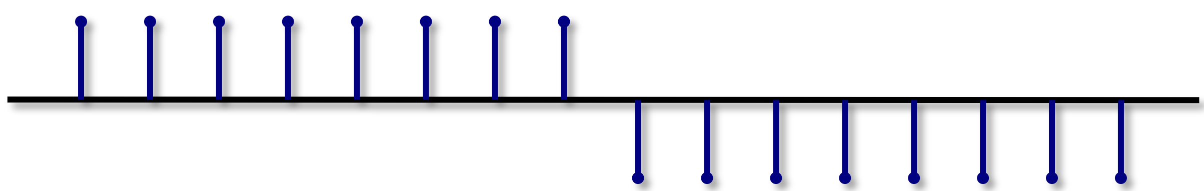

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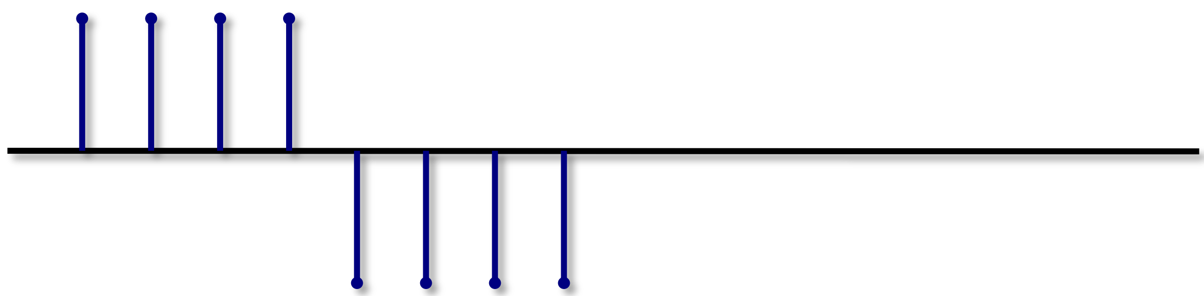

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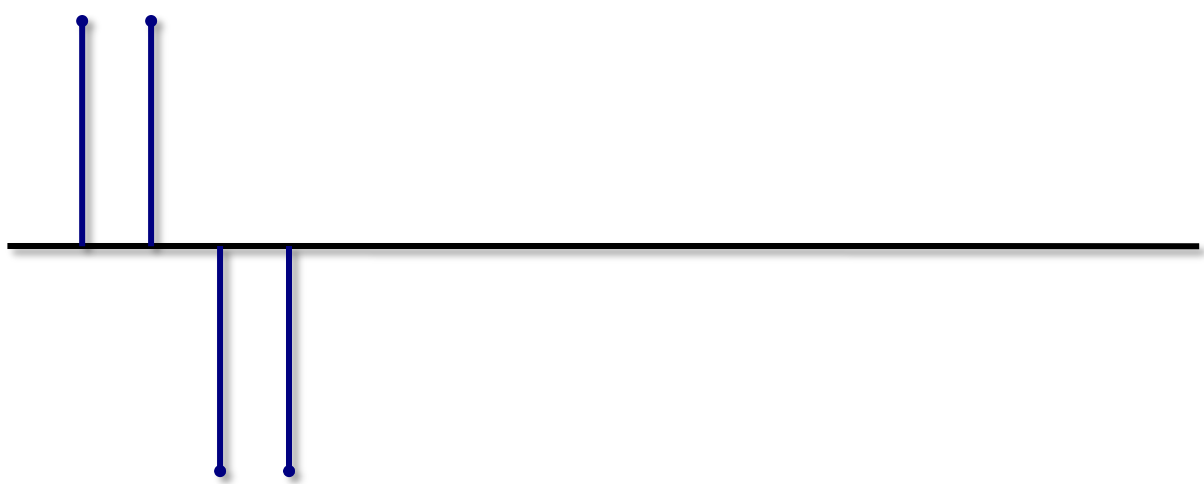

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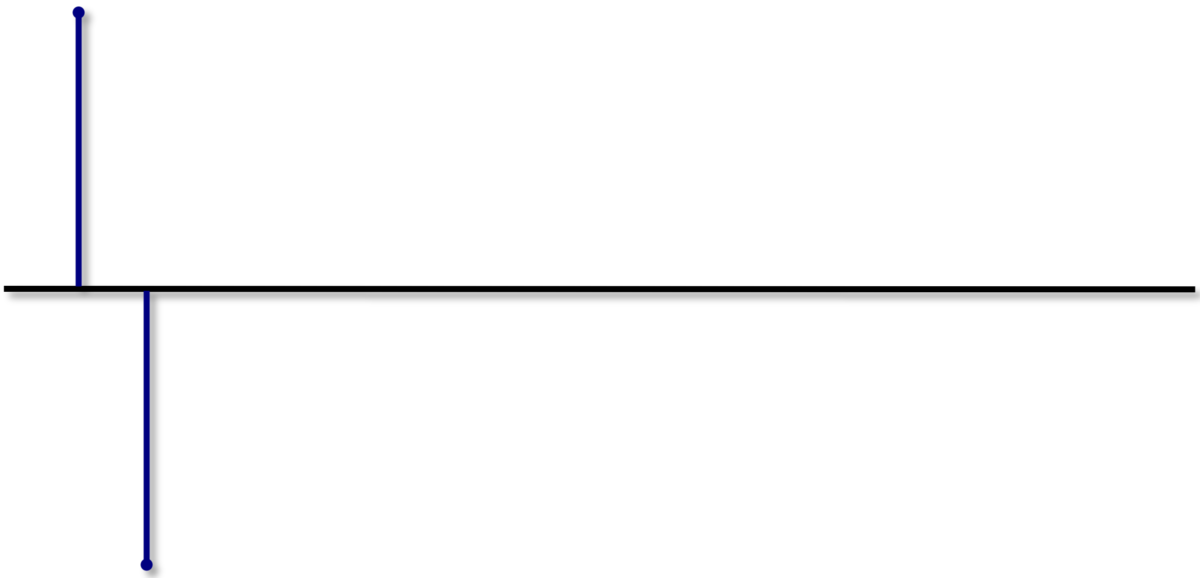

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Recursive Bisection

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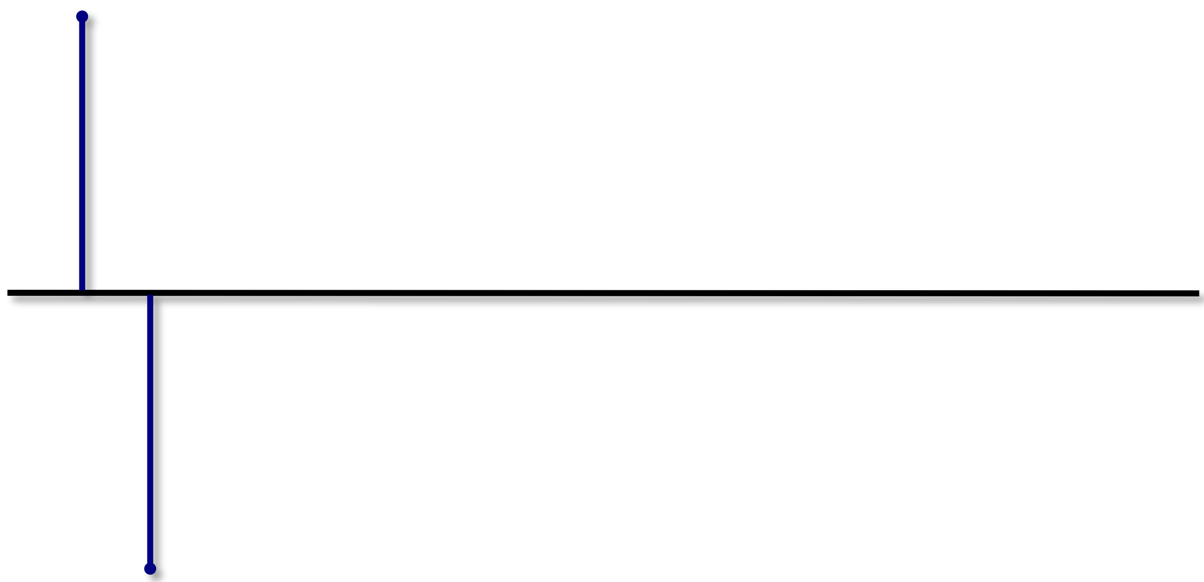
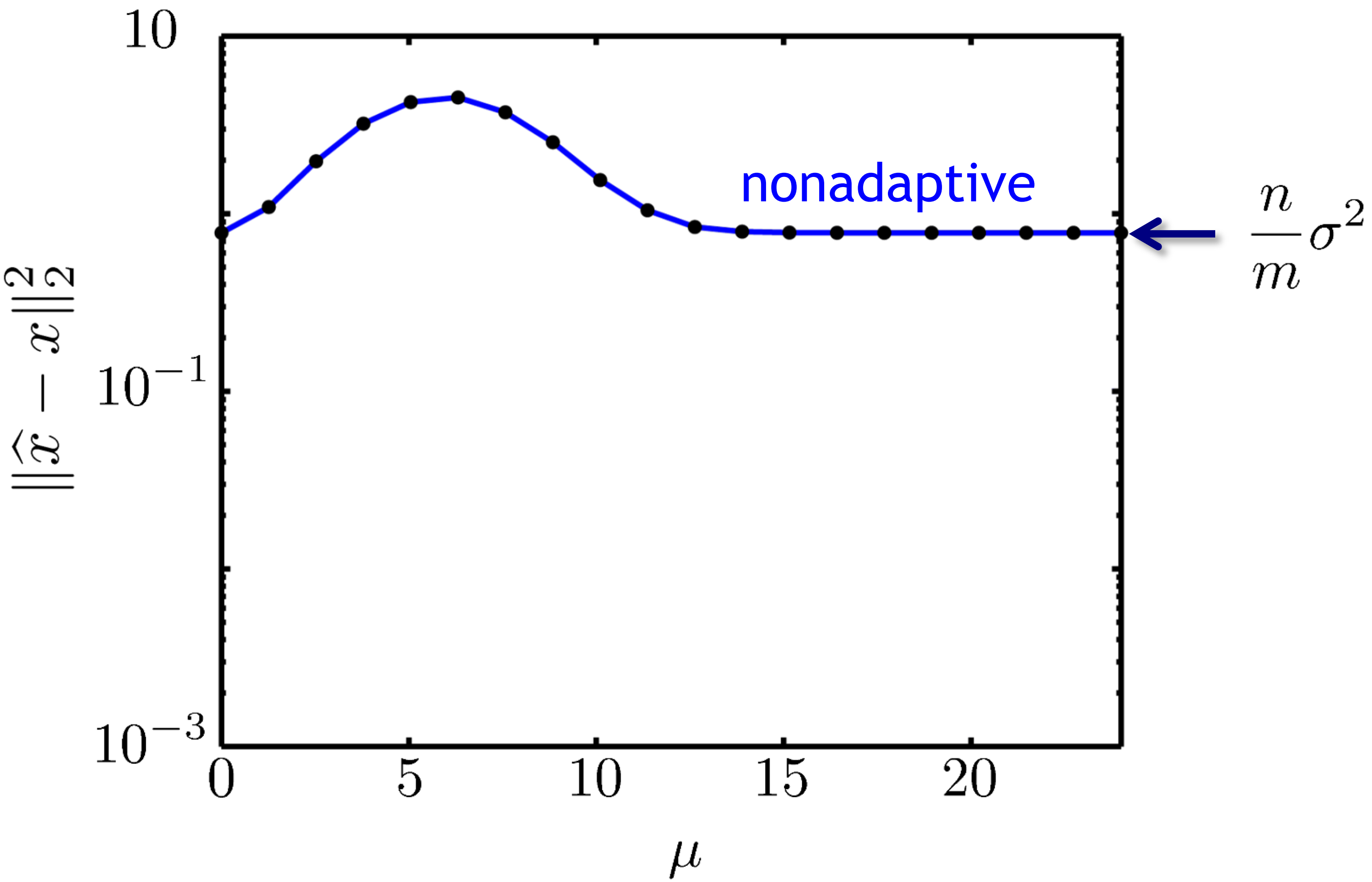
- in each stage, use measurements to decide if the nonzero is in the left or right half of the "active set"
- after subdividing $\log n$ times, return support



Experimental Results



$n = 512$

$m = 256$

[Arias-Castro, Candès, and Davenport - 2011]

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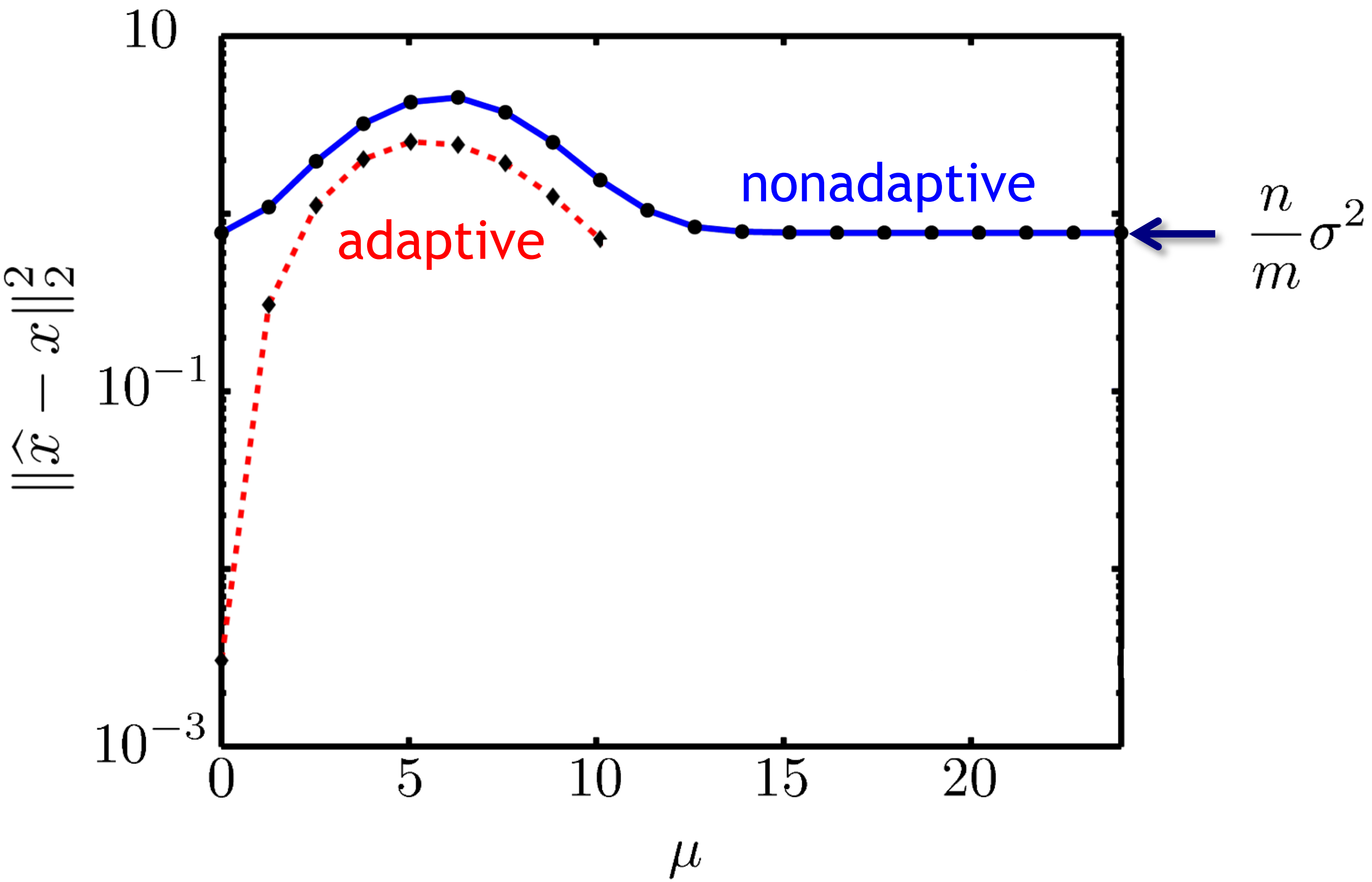

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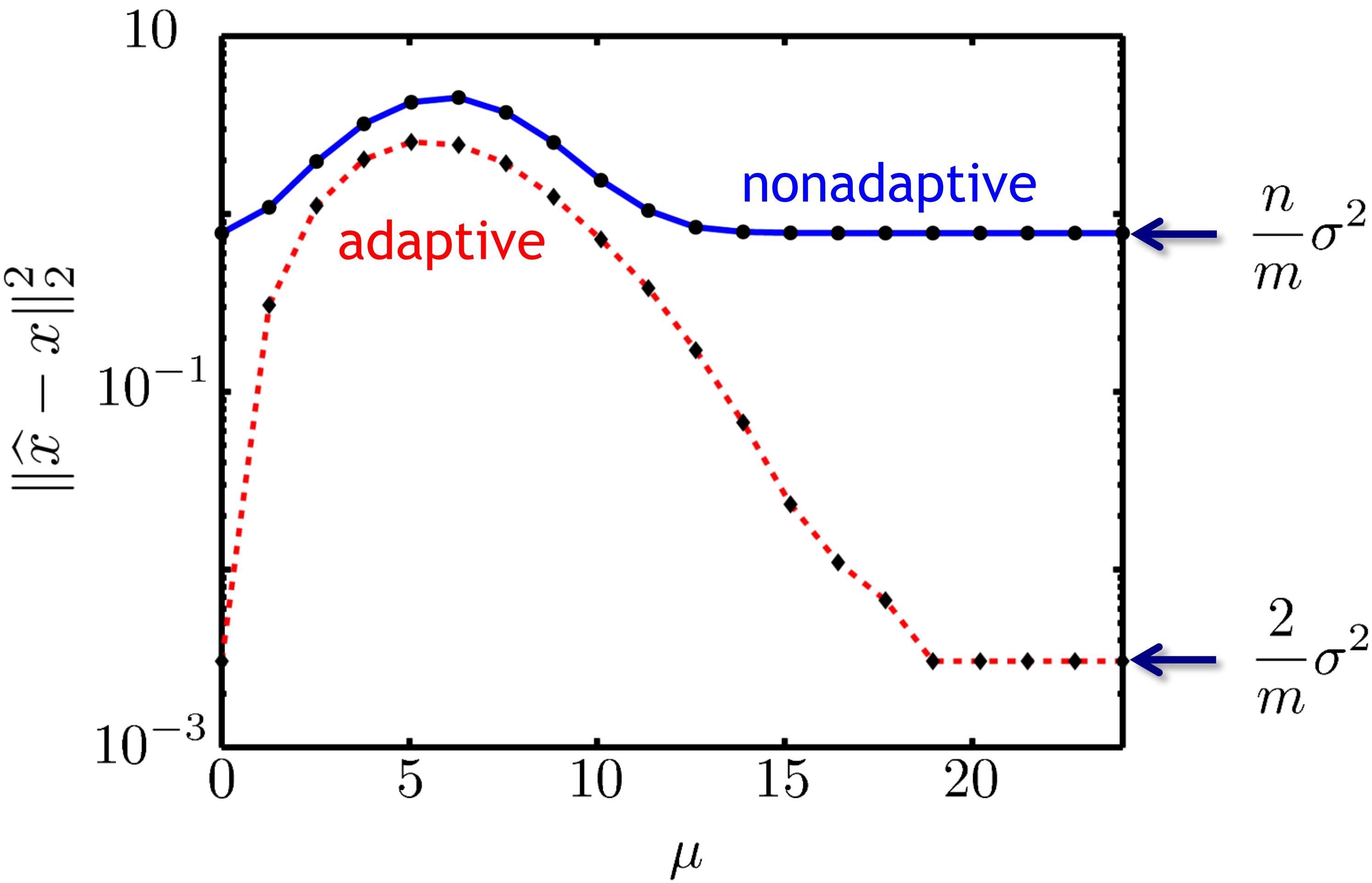

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- By iteration and/or divide-and-conquer approaches, can easily generalize this method to k-sparse vectors

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- Adaptivity doesn’t *always* help
- But when it does, the benefits are *overwhelming*
- Current techniques are not practical in many important situations
  - sensing process is typically constrained in some way
  - how to adapt the sensing process without violating these constraints?
  - how to retain the simple computational complexity of the decisions made at each step?