Compressive Binary Search

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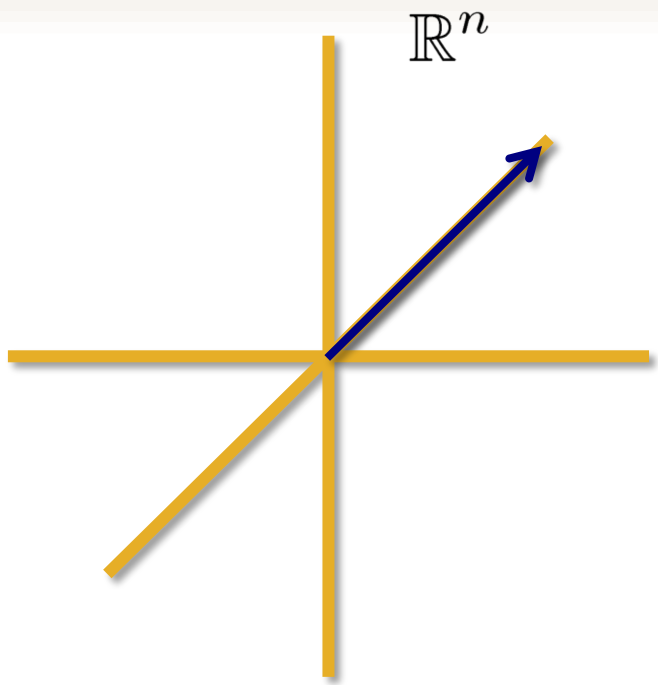
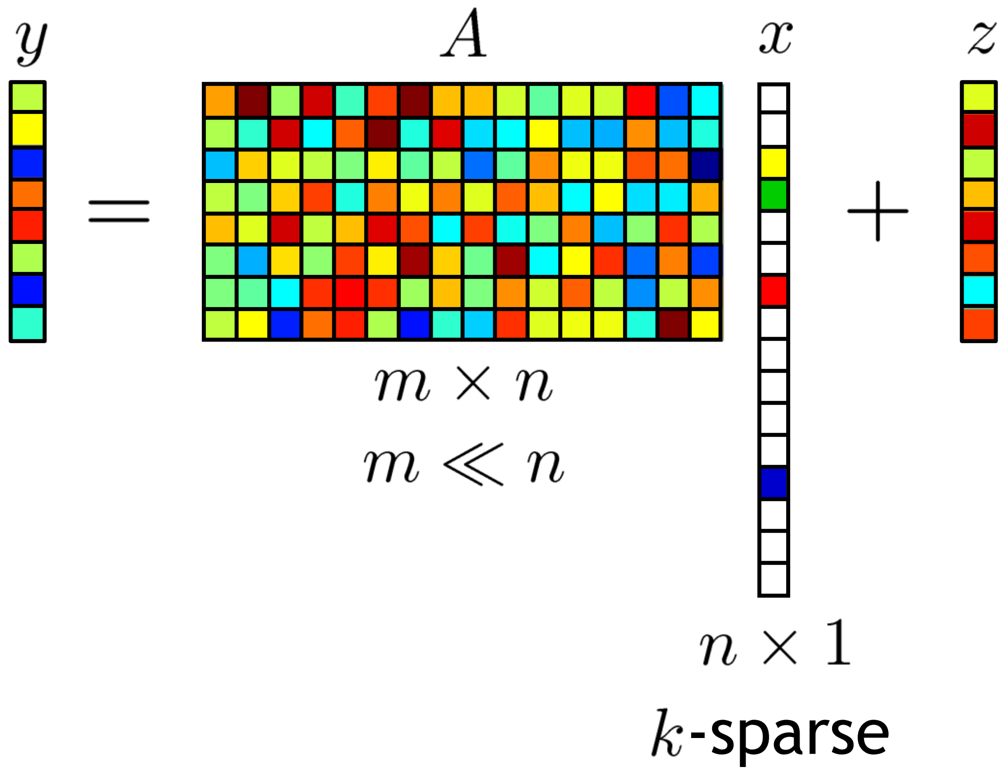
Finding a Needle in a Haystack

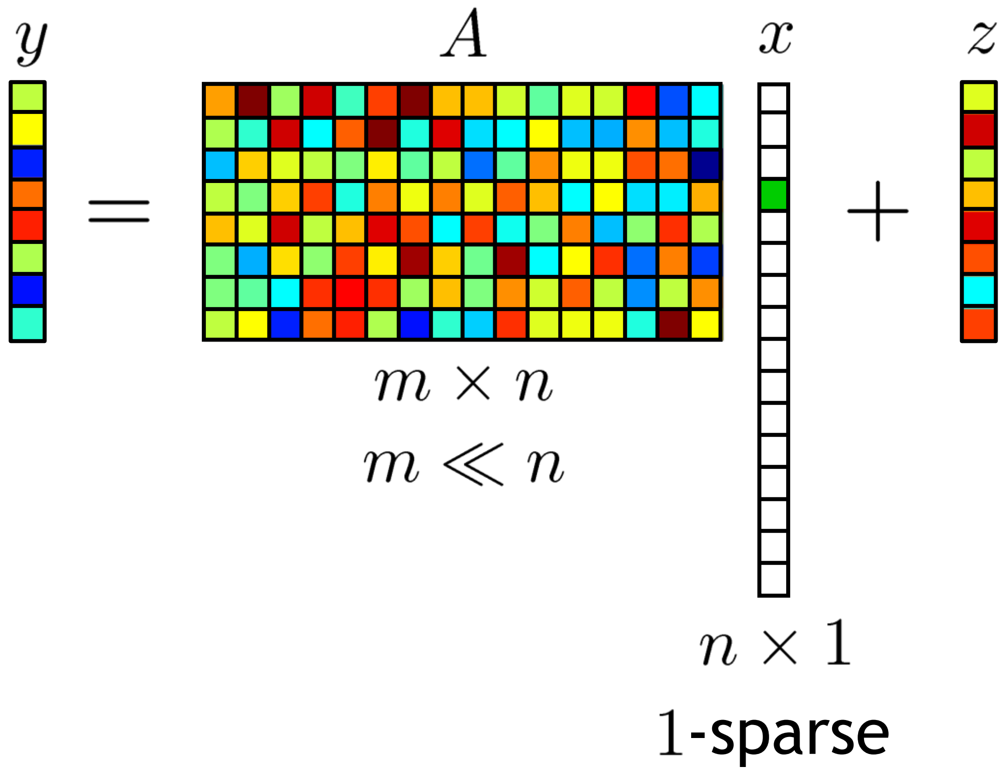


How many linear measurements
do we need to find the needle?

Compressive Sensing



Compressive Sensing



Measurement Model

$$y_i = \langle a_i, x \rangle + z_i$$

$$x_{j^*} = \mu \qquad \|a_i\|_2 = 1 \qquad z_i \sim \mathcal{N}(0, 1)$$

Suppose we can take m measurements.
How should we pick the a_i ?

How large does μ need to be?

Compressive Binary Search

Suppose for simplicity that $\mu > 0$

- split measurements into $s_{\max} = \log_2 n$ stages
- in each stage, use m_s measurements to decide if the nonzero is in the left or right half of the "active set"
- after stage $s_{\max}$, return estimate of location

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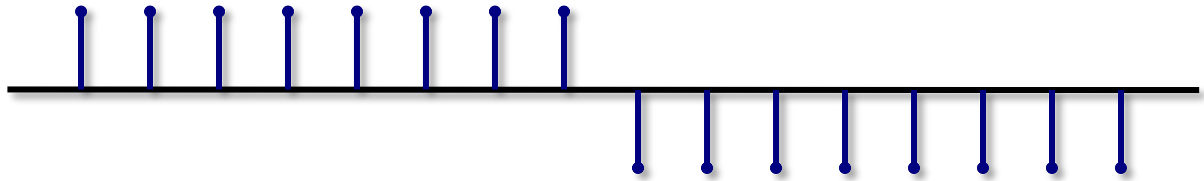

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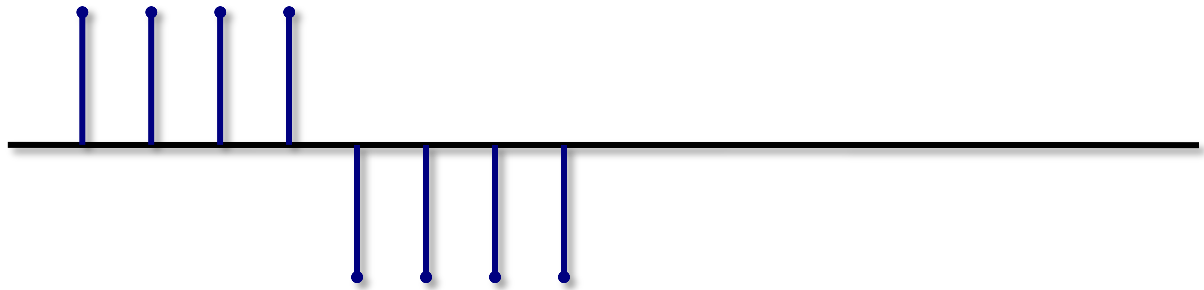

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Probability of Failure

Allocate m_s measurements to stage s

$$m_s = \lceil (m - s_{\max}) 2^{-s} \rceil$$

One can check that $\sum_s m_s \leq m$ and that if $m \geq 2 \log_2 n$, then at each stage

$$\mathbb{P}_e^{(s)} \leq \frac{1}{2} \exp\left(- \frac{\mu^2 m}{8n} \right)$$

Thus the total probability of failure is bounded by

$$\mathbb{P}_e \leq \frac{\log_2 n}{2} \exp\left(- \frac{\mu^2 m}{8n} \right)$$

How Large Must μ Be?

- Compressive binary search

$$\mu \geq \sqrt{8(n/m)\log\log_2 n}$$

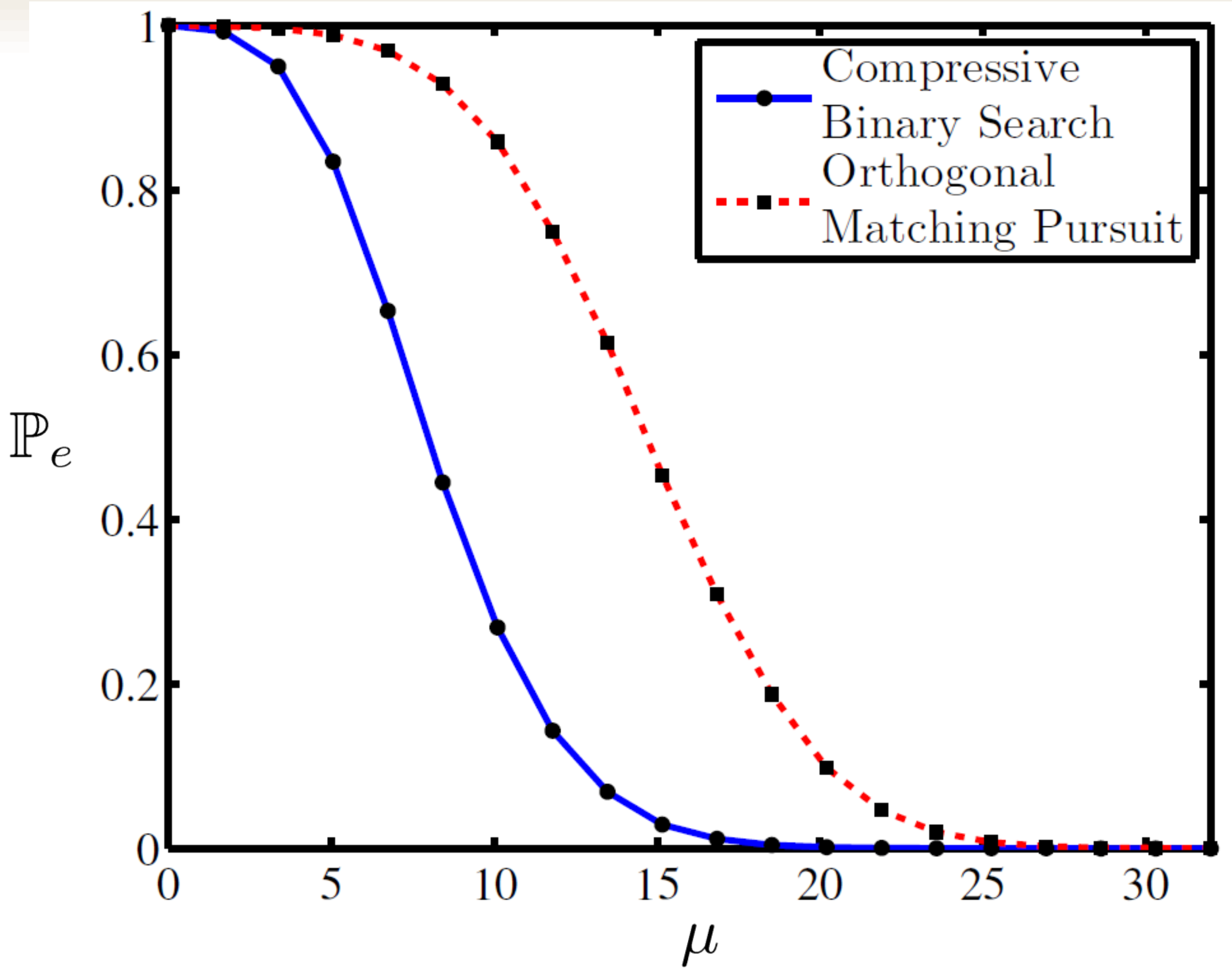
$$\Downarrow$$

$$\mathbb{P}_e \to 0$$

- Nonadaptive compressive sensing

$$\mu \geq C\sqrt{(n/m)\log n}$$

Experimental Results



Room For Improvement?

Is it possible to find substantially weaker spikes using some other method?

Theorem

Under the uniform prior, any procedure for finding the location of the nonzero will fail with probability

$$\mathbb{P}_e \geq \frac{1}{2}\left(1 - \mu\sqrt{m/n}\right).$$

Therefore, to reliably locate the nonzero we must have

$$\mu \geq C\sqrt{n/m}.$$

Testing Left Versus Right

- $\mathcal{H}_1$: the nonzero is in the "left" half
- $\mathcal{H}_2$: the nonzero is in the "right" half

Consider any test T for choosing between $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\mathbb{P}\left(T \text{ fails}\right) \geq \frac{1}{2}\left(1 - \mu\sqrt{m/n}\right)$$

Follows from the facts that

- $\mathbb{P}\left(T \text{ fails}\right) \geq \frac{1}{2}\left(1 - \|\mathbb{P}_2 - \mathbb{P}_1\|_{\mathrm{TV}}\right)$
- $\|\mathbb{P}_2 - \mathbb{P}_1\|_{\mathrm{TV}}^2 \leq \frac{\mu^2 m}{n}$

Key Ideas in Proof

Pinsker's Inequality

$$\|\mathbb{P} - \mathbb{Q}\|_{\mathrm{TV}} \leq \sqrt{K(\mathbb{P}, \mathbb{Q})/2}$$

Define $\mathbb{P}_0$ as the distribution of $y_1, \ldots, y_m$ when $x = 0$

$$\begin{aligned}
\|\mathbb{P}_2 - \mathbb{P}_1\|_{\mathrm{TV}}^2 &\leq 2\|\mathbb{P}_0 - \mathbb{P}_1\|_{\mathrm{TV}}^2 + 2\|\mathbb{P}_0 - \mathbb{P}_2\|_{\mathrm{TV}}^2 \\
&\leq K(\mathbb{P}_0, \mathbb{P}_1) + K(\mathbb{P}_0, \mathbb{P}_2) \\
&\leq \frac{\mu^2}{n} \sum_i \mathbb{E}_0 \sum_{j \in J_1} a_{ij}^2 + \frac{\mu^2}{n} \sum_i \mathbb{E}_0 \sum_{j \in J_2} a_{ij}^2 \\
&= \frac{\mu^2}{n} \sum_i \sum_j \mathbb{E}_0 a_{ij}^2 = \frac{\mu^2 m}{n}
\end{aligned}$$

Implications for Estimation

For any estimator $\widehat{x}$, there exist x for which

$$\frac{1}{n}\mathbb{E}\|\widehat{x} - x\|^2 \geq \frac{1}{27}\frac{1}{m}$$

$$\begin{aligned} \mathbb{E}\,\|\widehat{x} - x\|_2^2 &\geq \frac{\mu^2}{2}\mathbb{P}_e \\ &\geq \frac{\mu^2}{4}\left(1 - \mu\sqrt{m/n}\right) \end{aligned}$$

$$\mu = \frac{2}{3}\sqrt{n/m} \quad \Longrightarrow \quad \geq \frac{1}{27}\frac{n}{m}$$

Conclusions

- Compressive binary search can successfully identify the location of the nonzero when

$$\mu \geq \sqrt{8(n/m)\log\log_2 n}$$

- Malloy and Nowak improve this to

$$\mu \geq C'\sqrt{n/m}$$

  at the cost of a slightly worse constant

- No method can succeed unless

$$\mu \geq C\sqrt{n/m}$$

- It remains to generalize to k-sparse vectors