Compressive Imaging: Theory and Practice

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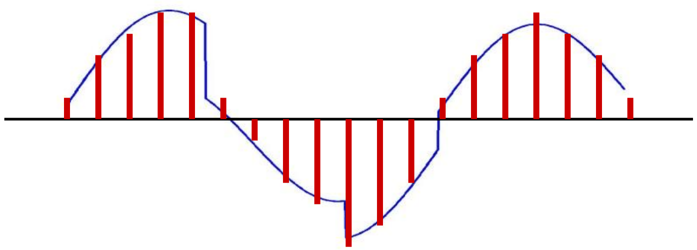
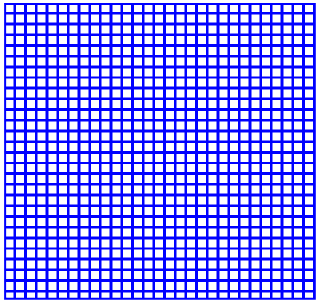
Digital Revolution

Digital Acquisition

- Foundation: ***Shannon sampling theorem***

Must sample at 2x highest frequency
of the signal (Nyquist rate)





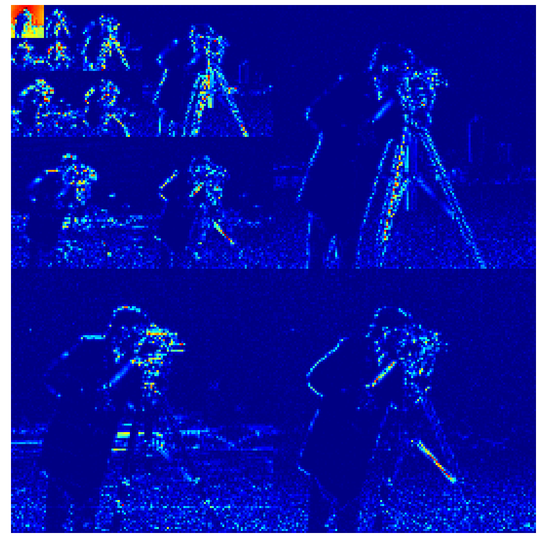
- Time: A/D converters, receivers, …
- Space: cameras, imaging systems, …

- High-frequency content = **lots of samples…**

Sparsity

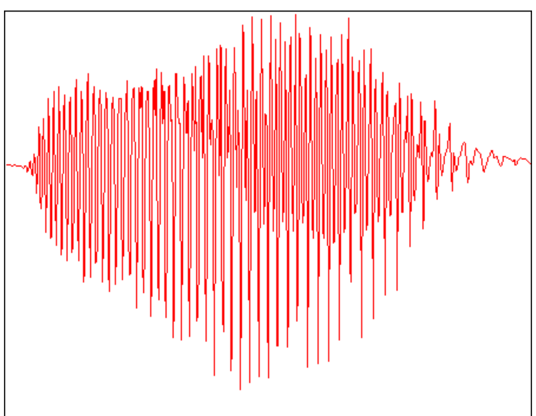
- Many signals can be ***compressed*** in some representation/basis (Fourier, wavelets, …)

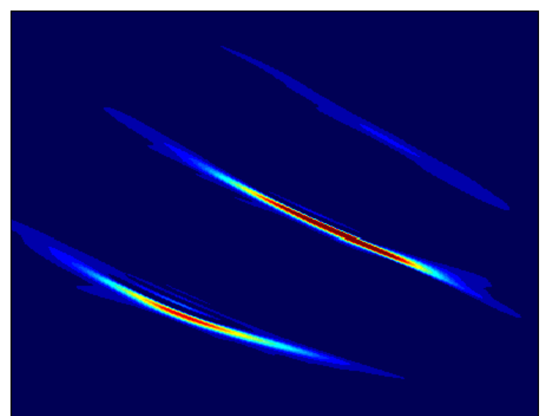
N
pixels



$K \ll N$
large
wavelet
coefficients

N
wideband
signal
samples

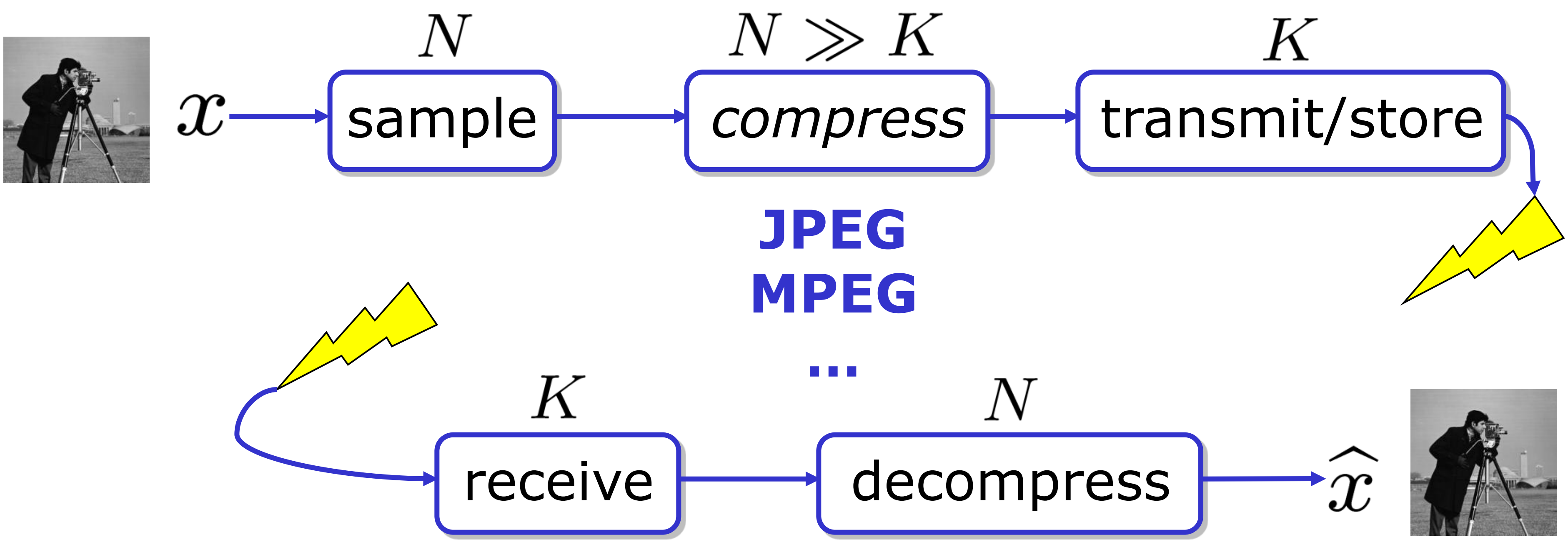




$K \ll N$
large
Gabor
coefficients

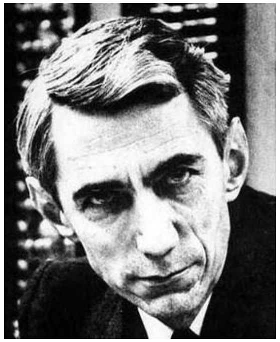
Sensing by *Sampling*

- Standard paradigm for digital data acquisition
  - ***sample*** data (ADC, digital camera, …)
  - ***compress*** data (signal-dependent, nonlinear)



- Sample-and-compress paradigm is ***wasteful***
  - samples cost $$$ and/or time

From Samples to *Measurements*



- Shannon was a *pessimist*
  - worst case bound for *any* bandlimited signal

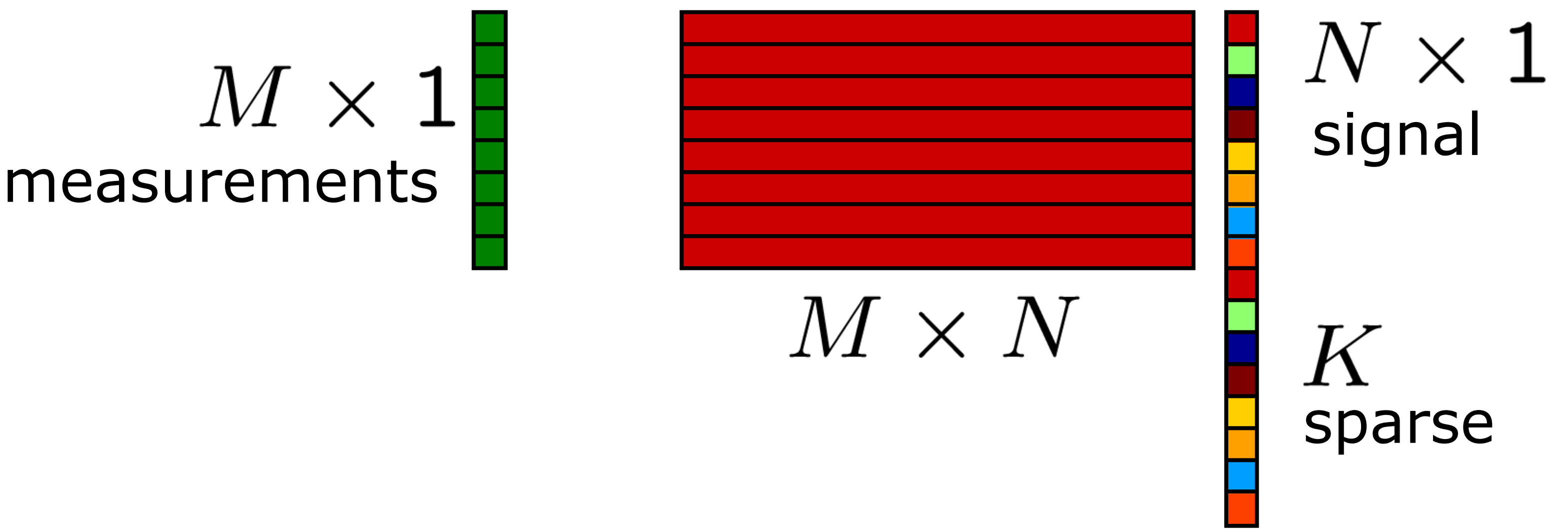
- ***Compressive sensing***
  [Candes, Romberg, Tao; Donoho – 2004]
  - generalize “samples” to linear “measurements”
  - incorporate prior knowledge about the signal (sparsity)

Goal: Take as few measurements as possible while retaining the ability to accurately recover the signal from the measurements.

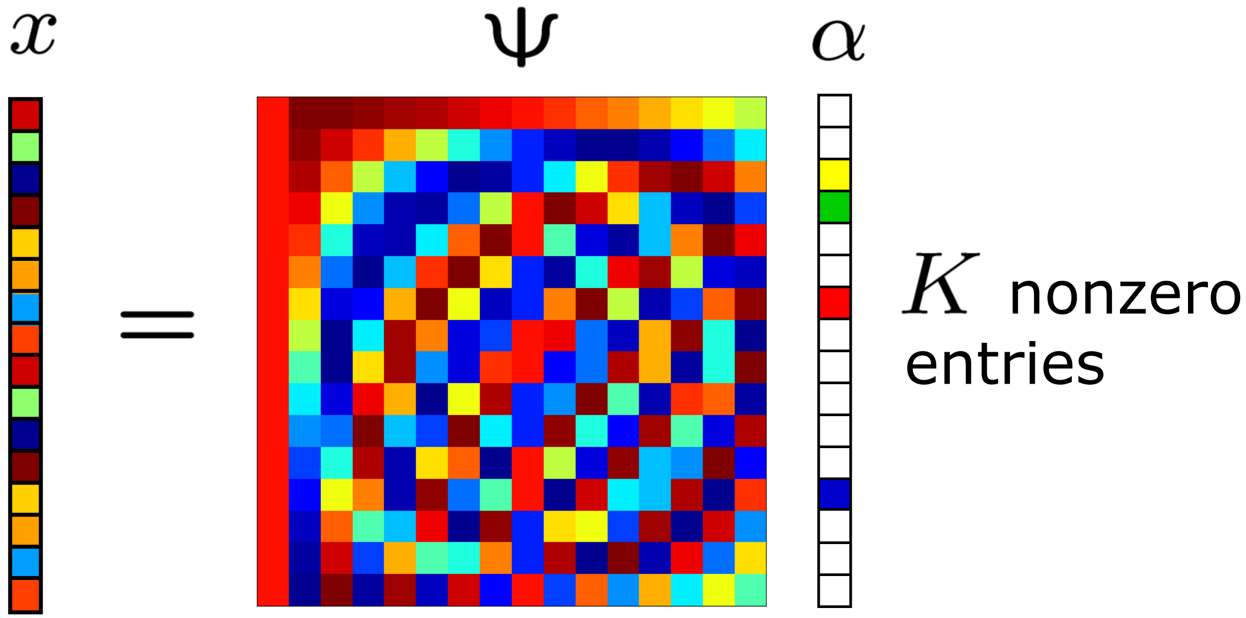
Compressive Sensing

Replace samples with *linear measurements*

$$y = \Phi x$$



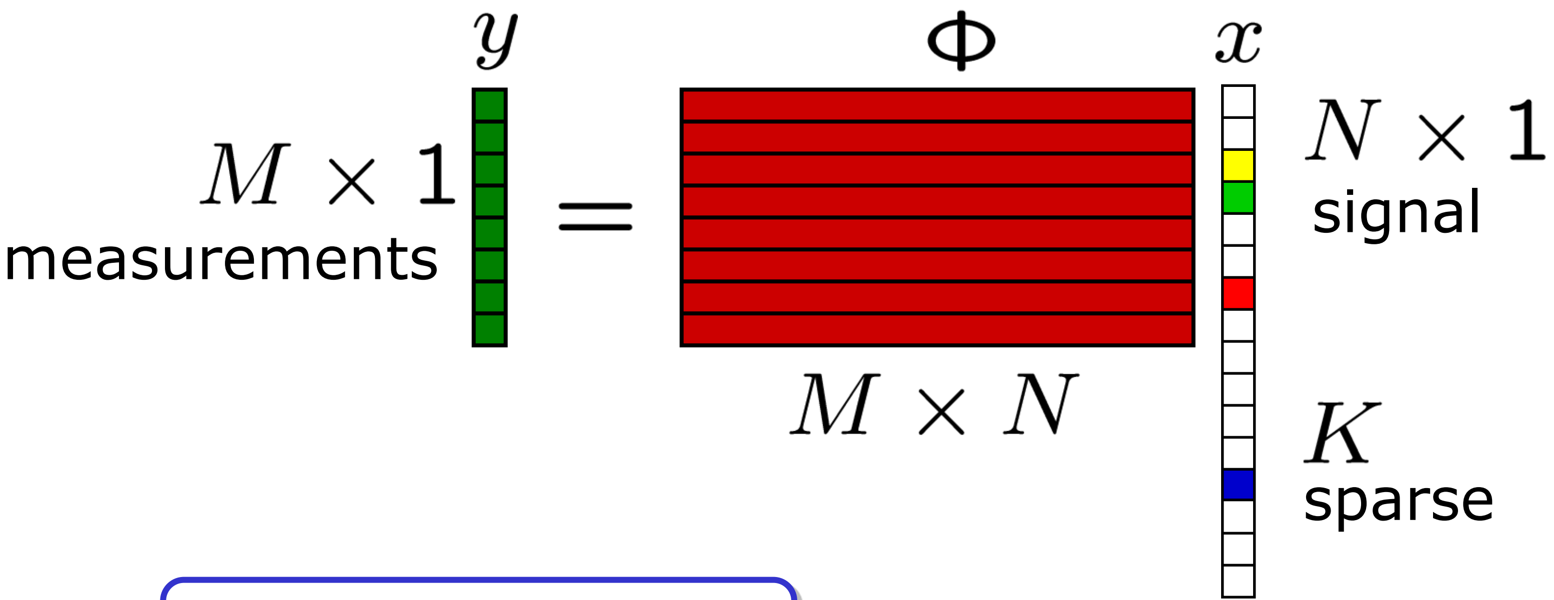
Sparsity



- For now: Assume $\Psi = I$

Compressive Sensing

Replace samples with *linear measurements*

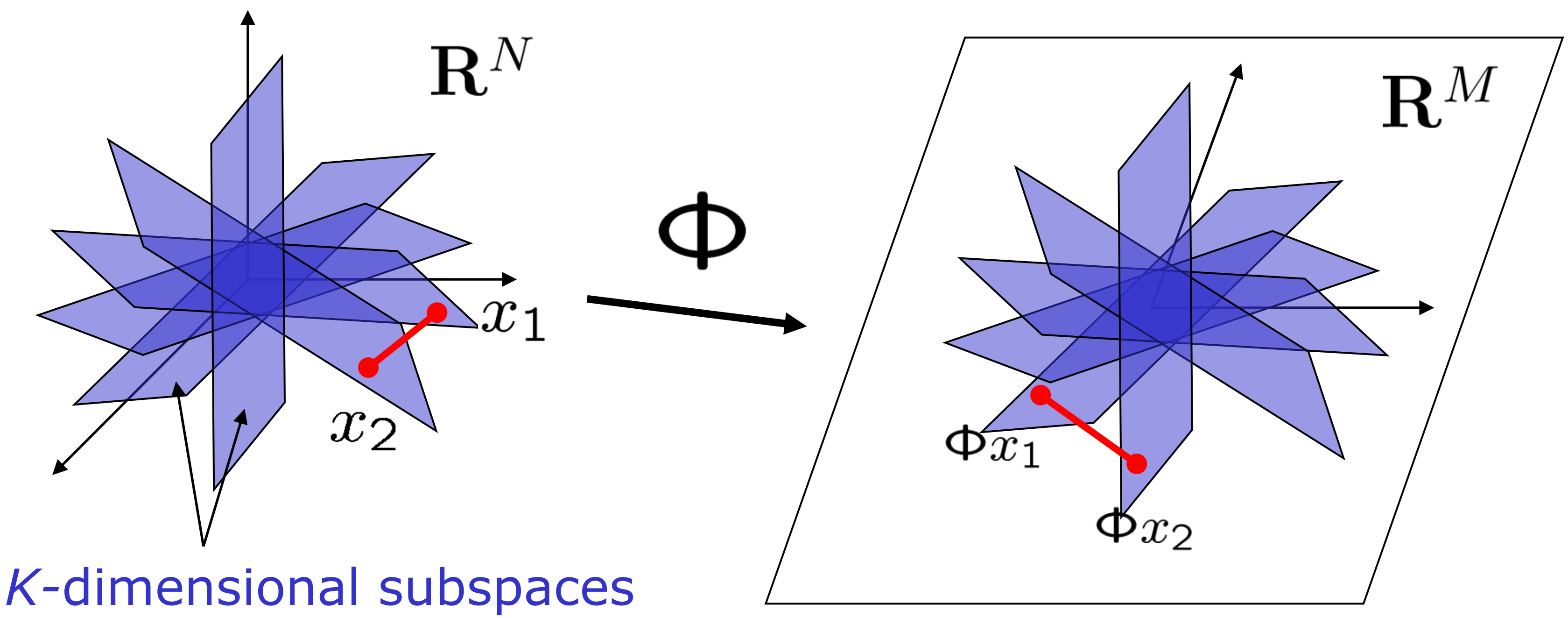


$$\boxed{K < M \ll N}$$

Restricted Isometry Property (RIP)

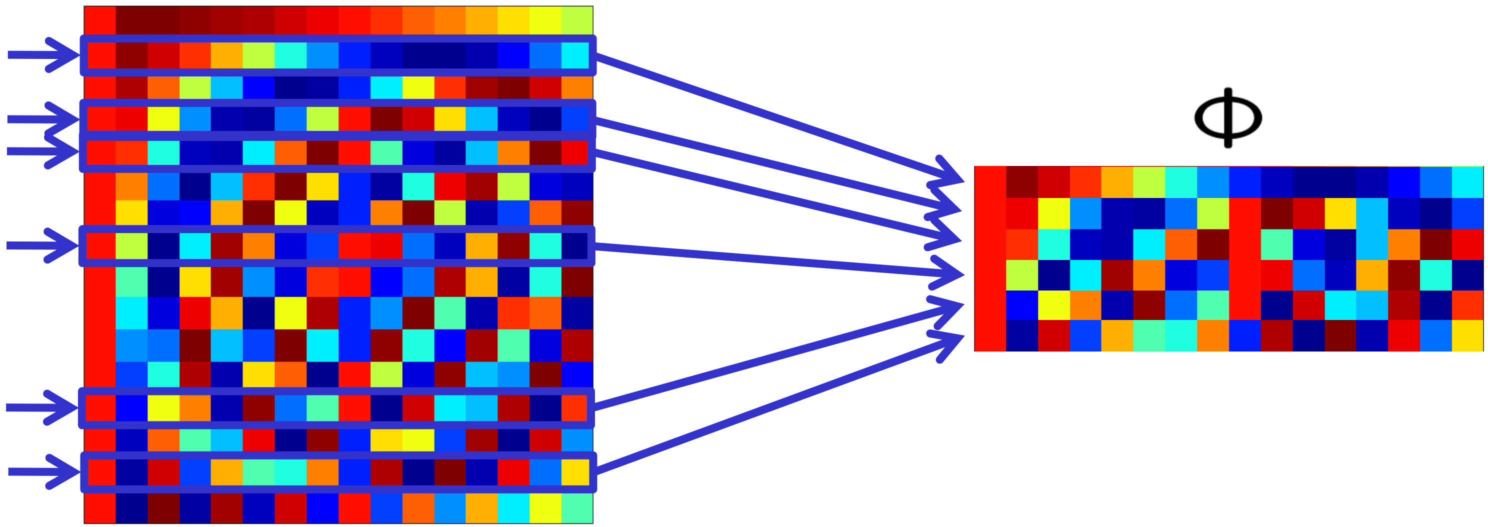
- Preserve the structure of sparse signals
- For all *K*-sparse x_1 and x_2

$$(1-\delta_{2K}) \leq \frac{\|\Phi x_1 - \Phi x_2\|_2^2}{\|x_1 - x_2\|_2^2} \leq (1+\delta_{2K})$$



RIP Matrix: Option 1

- Random Fourier submatrix:



- If the rows are selected at random with

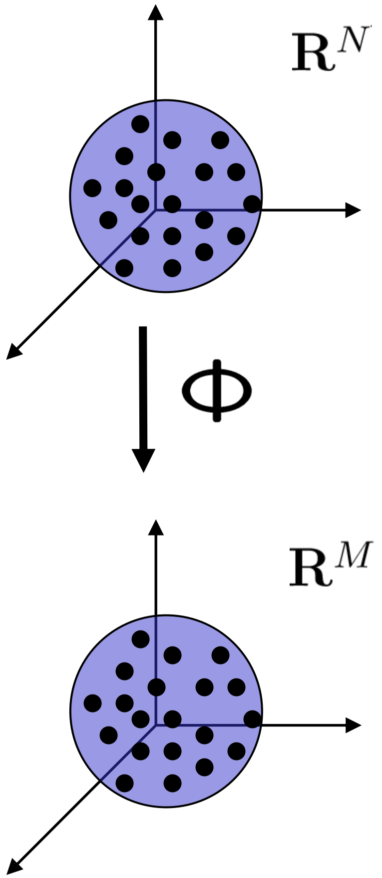
$$M \geq CK \log^4(N)$$

then with high probability, Φ will satisfy the RIP [Candes and Tao]

RIP Matrix: Option 2

- Pick Φ at *random*
  - i.i.d. Gaussian
  - i.i.d. Bernouli
  - *any* bounded random variable

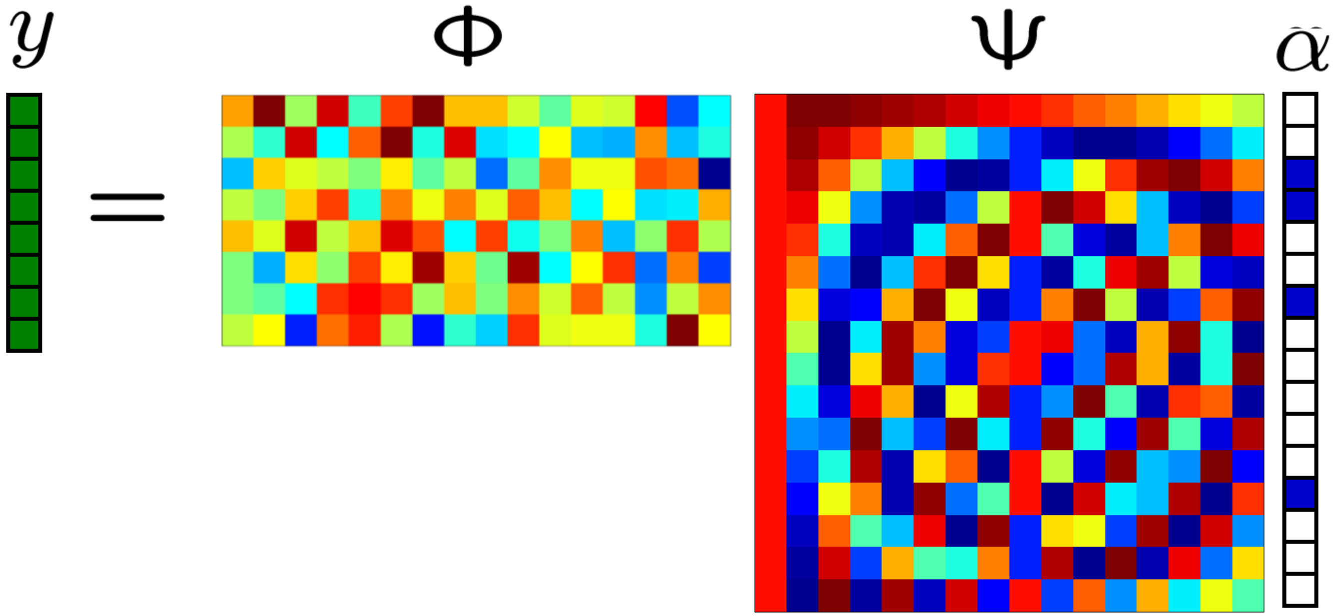
$$M \geq CK \log(N/K)$$



- Proof relies on concentration of measure [Baraniuk, Davenport, DeVore, Wakin]
  - fix a $2K$-dimensional subspace
  - pick a finite sampling of points on the sphere
  - repeat for all $\binom{N}{2K}$ subspaces
  - argue that Φ preserves the norm of each point
  - extend from point set to entire sphere

Universality

- Random matrix will work with ***any*** fixed orthonormal basis (with high probability)

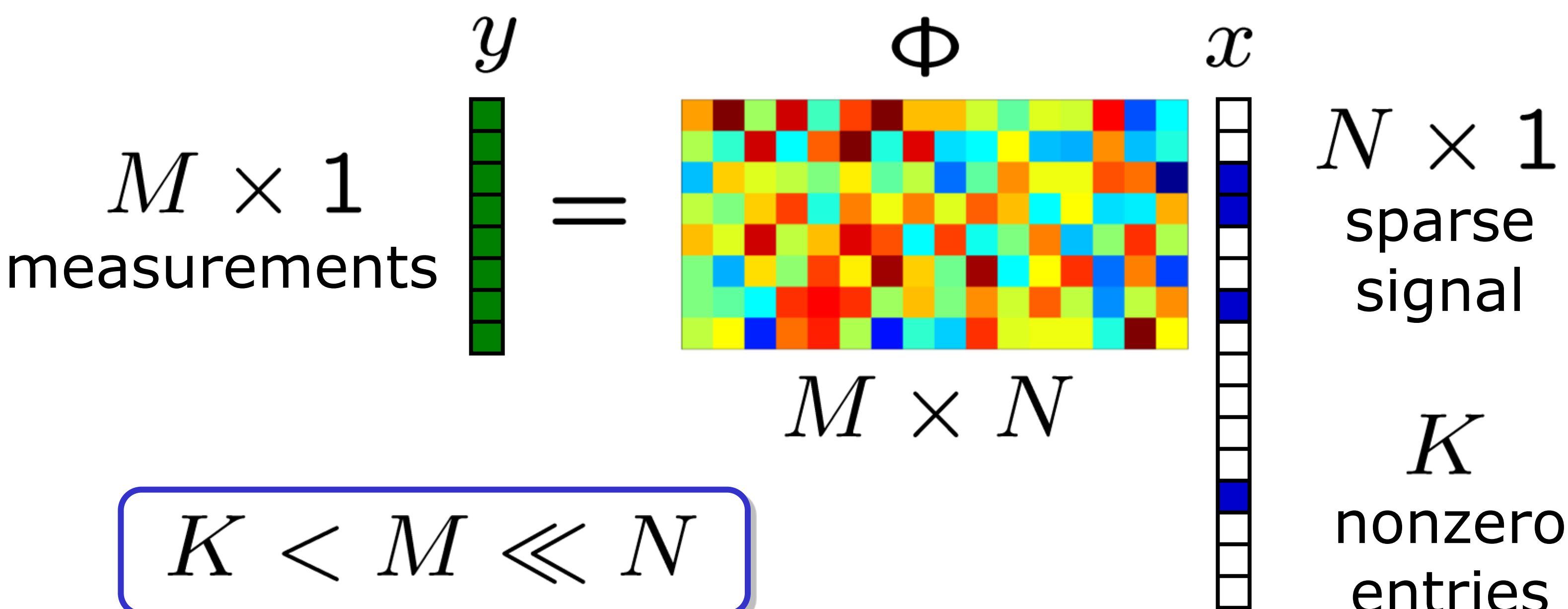


Signal Recovery

- Reconstruction/decoding:

$$\boxed{\begin{array}{c}\text{given } y = \Phi x \\ \text{find } x\end{array}}$$

ill-posed
inverse problem



$$\boxed{K < M \ll N}$$

Signal Recovery

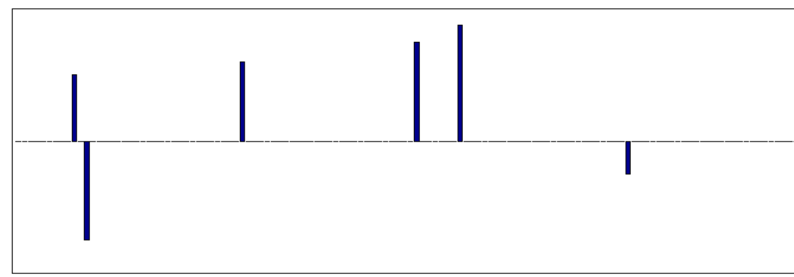
- Reconstruction/decoding:
  (ill-posed inverse problem)

given $y = \Phi x$
find x

- $\boldsymbol{L_2}$: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_2 \longrightarrow \widehat{x} = (\Phi^T\Phi)^{-1}\Phi^T y$

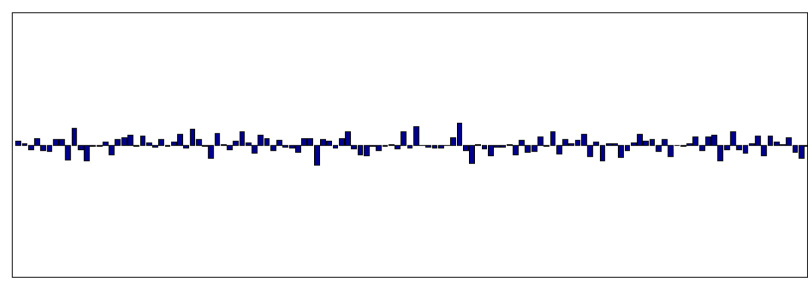
- Fast, but *wrong*

x



- Solution is
  almost never sparse

$\widehat{x}$



Signal Recovery

- Reconstruction/decoding:
  (ill-posed inverse problem)

given $y = \Phi x$
find x

- ***L*$_2$**: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_2$
- ***L*$_0$**: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_0$

number of nonzero entries

- Correct, but *slow* (NP-Hard)

Signal Recovery

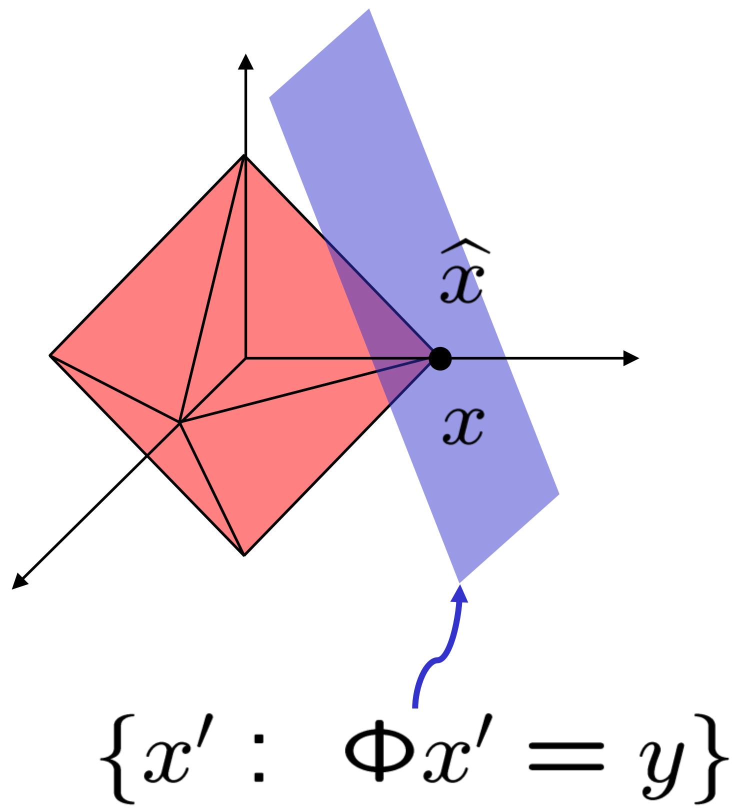
- Reconstruction/decoding:
  (ill-posed inverse problem)

given $y = \Phi x$
find x

- $\boldsymbol{L_2}$: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_2$
- $\boldsymbol{L_0}$: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_0$
- $\boldsymbol{L_1}$: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_1$ ← *linear program*
- If Φ satisfies the RIP, $\boldsymbol{L_1}$ gives same answer as $\boldsymbol{L_0}$

Why L_1 Works

$$\widehat{x} = \arg\min_{y=\Phi x'} \|x'\|_1$$



Recovery in Noise

- What about noise, or robustness to non-sparse signals?

$$y = \Phi x + e$$

$$\widehat{x} = \underset{\|y - \Phi x'\|_2 \leq \epsilon}{\arg\min} \|x'\|_1$$

$$\boxed{\|\widehat{x} - x\|_2 \leq C_0 \|e\|_2 + C_1 \frac{\|x - x_K\|_1}{\sqrt{K}}}$$

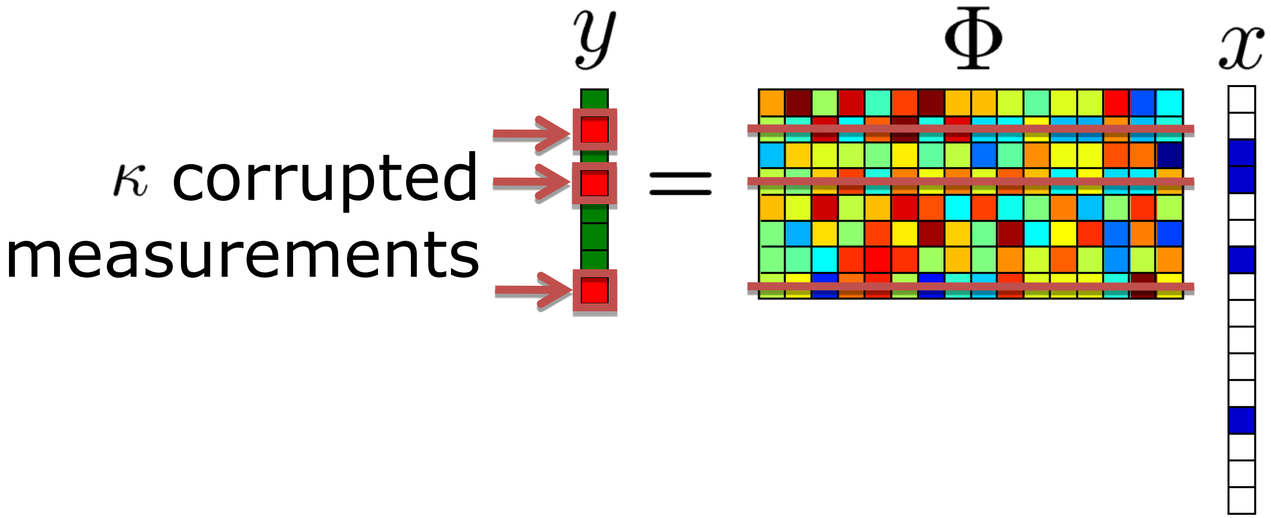
Compressive Sensing Hallmarks

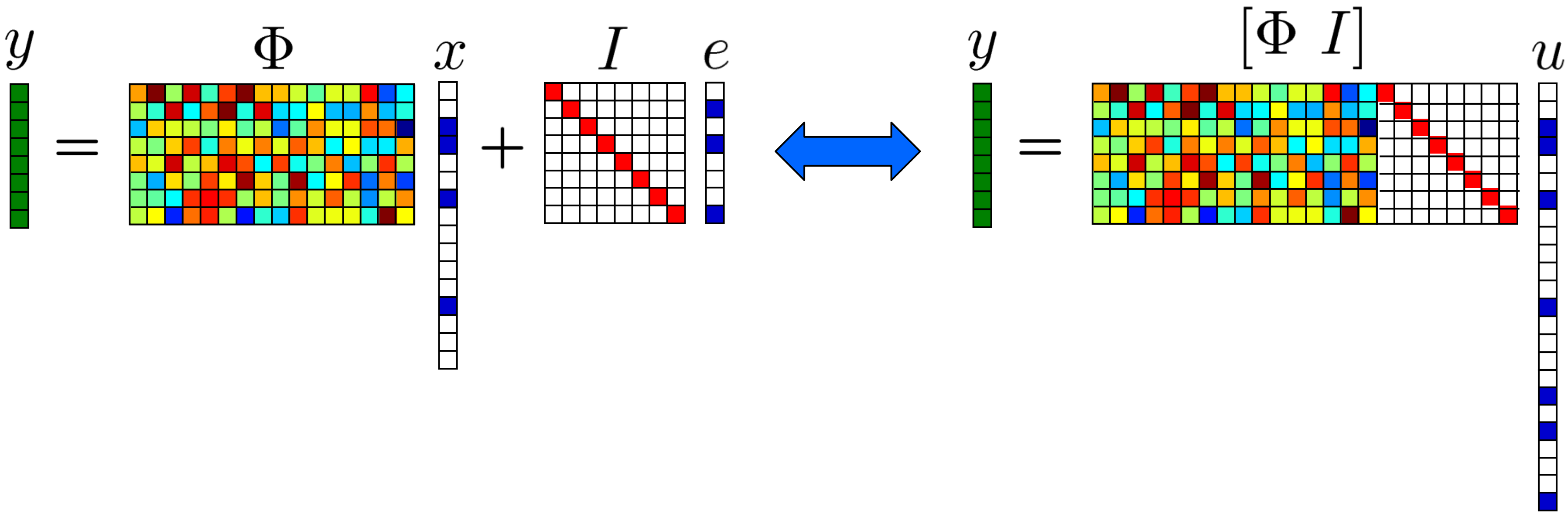
- Asymmetrical
  - no processing at encoder
  - significant processing at decoder

- Universal
  - random projections / hardware can be designed and used without prior knowledge of the sparsifying basis

- Democratic
  - each measurement carries the same amount of information
  - simple encoding
  - robust to measurement loss and quantization

Democracy and Sparse Noise





Justice Pursuit

$$\widehat{u} = \arg\min_{u} \|u\|_1$$

$$\text{s.t.} \quad y = [\Phi \; I]\, u$$

Theorem: If Φ is a subGaussian matrix with

$$M = O\left((K+\kappa)\log\left(\frac{N+M}{K+\kappa}\right)\right)$$

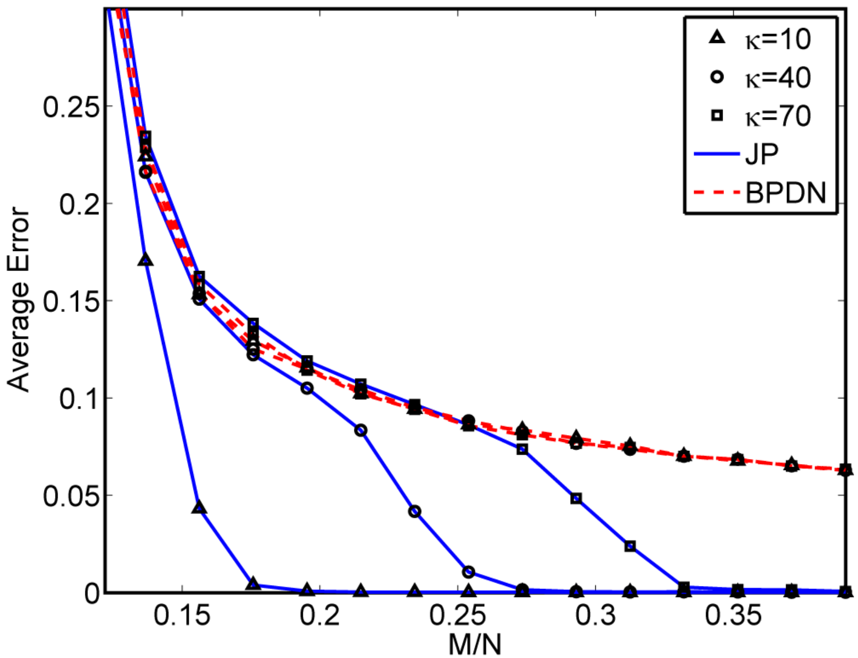
then $[\Phi \; I]$ satisfies the RIP of order $(K+\kappa)$ with probability at least $1 - 3e^{-CM}$.

[Laska, Davenport, Baraniuk]

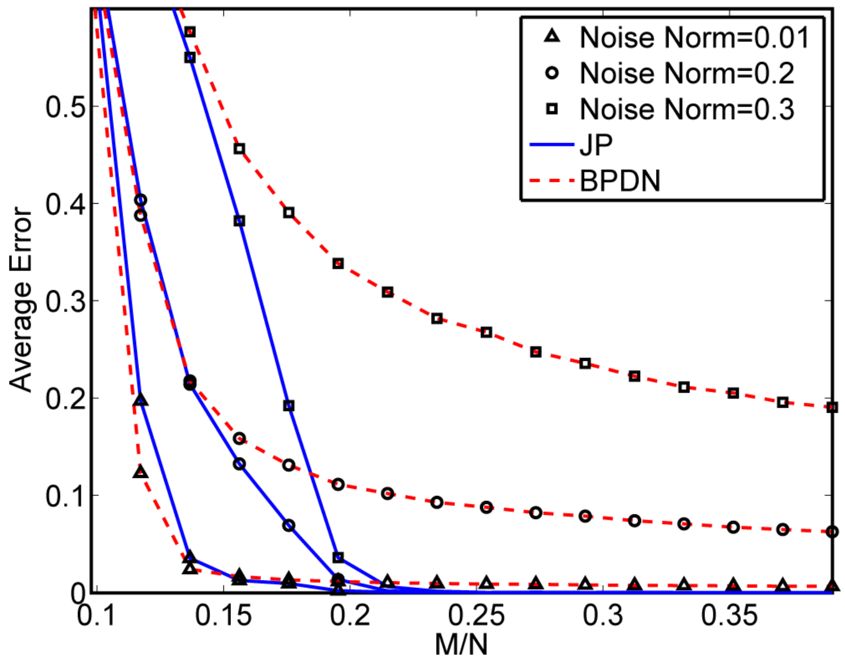
Justice Pursuit

- We can recover sparse signals *exactly* in the presence of *unbounded* sparse noise

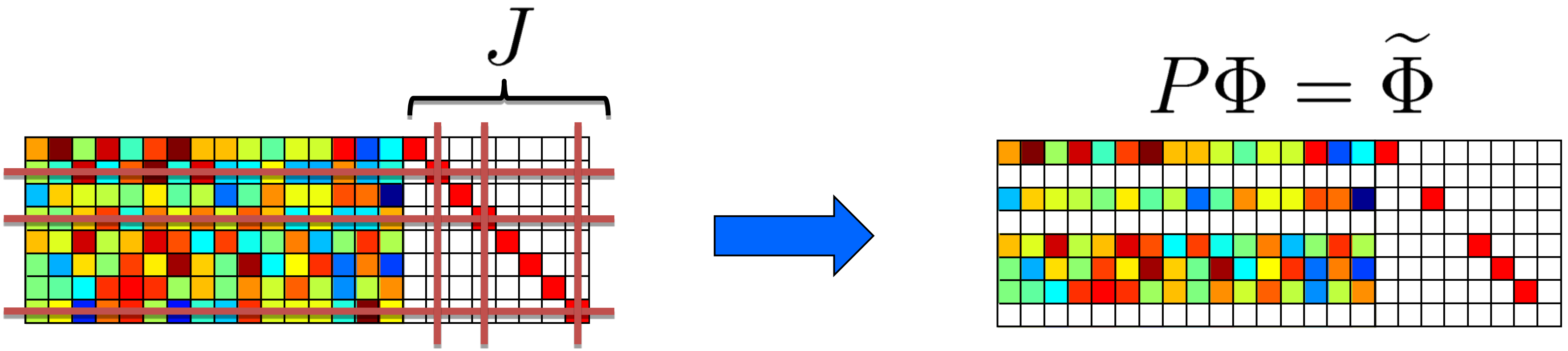
Fixed $\|e\|_2 = 0.1$



Fixed $\kappa = 10$



Justice and Democracy



- The fact that $[\Phi \;\; I]$ satisfies the RIP also implies that we can delete arbitrary rows of Φ and retain the RIP

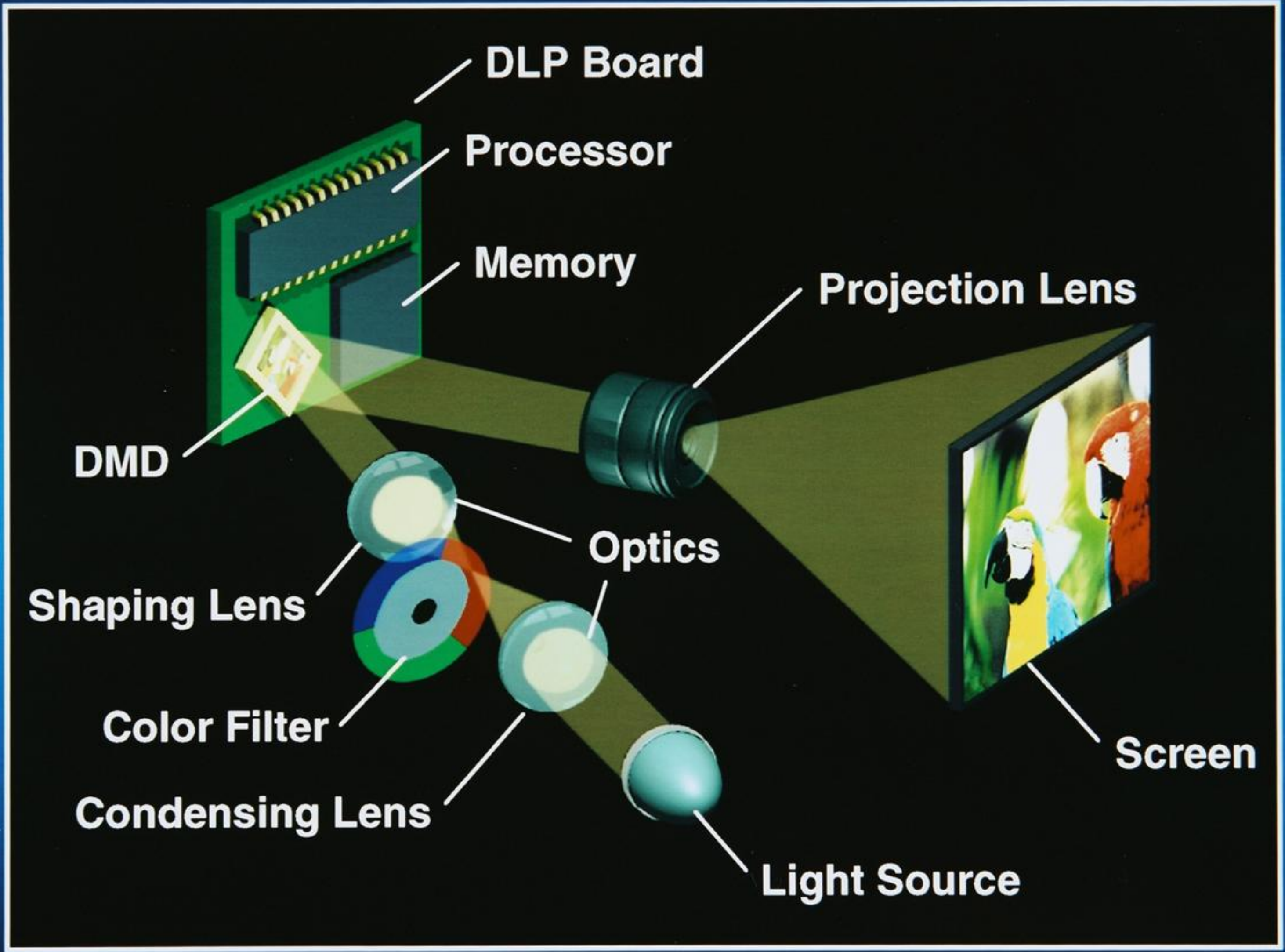
- Random matrices satisfy a *very strong* ***adversarial*** form of democracy

Compressive Imaging in Practice

- Tomography in medical imaging
  - each projection gives you a set of Fourier coefficients
  - fewer measurements mean
    - more patients
    - sharper images
    - less radiation exposure

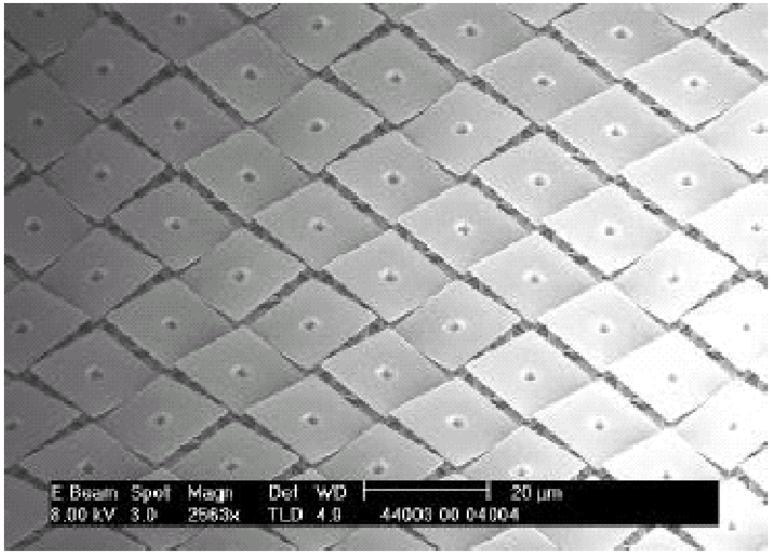
- Conventional imaging at non-visible wavelengths
  - cannot always build sensor arrays
  - raster scan takes time

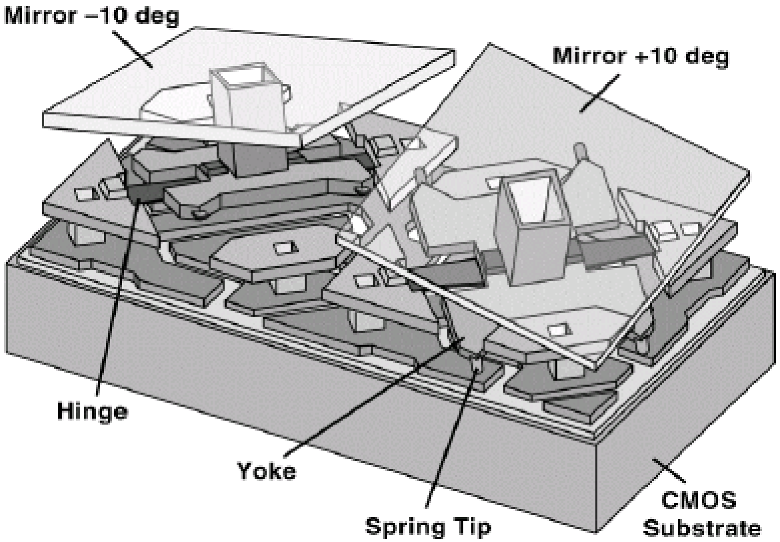
1 Chip DLP™ Projection



DLP™

TI Digital Micromirror Device





“Single-Pixel” Camera

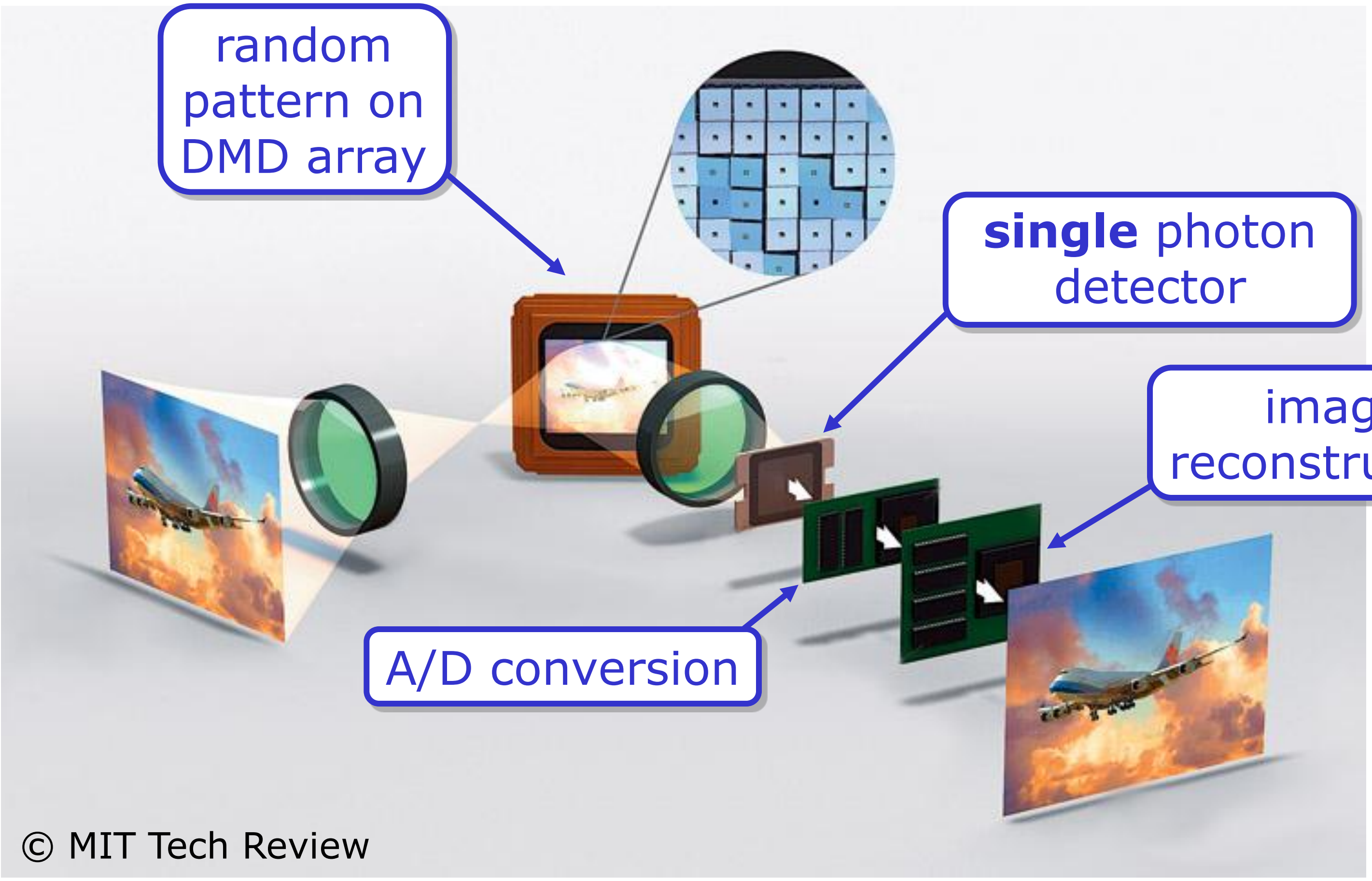
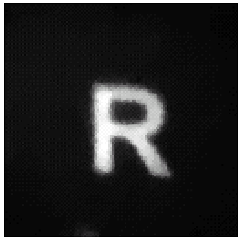
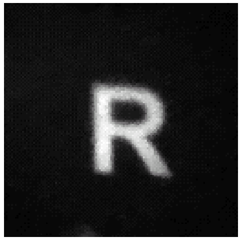


Image Acquisition

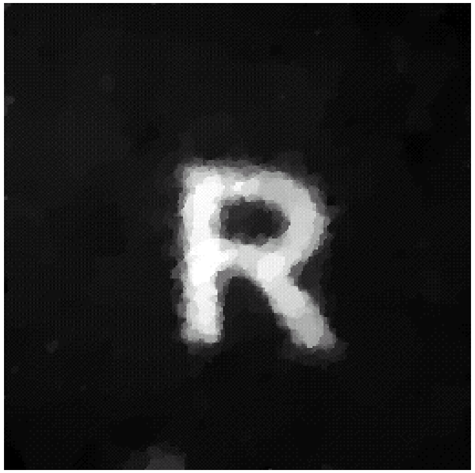
Original



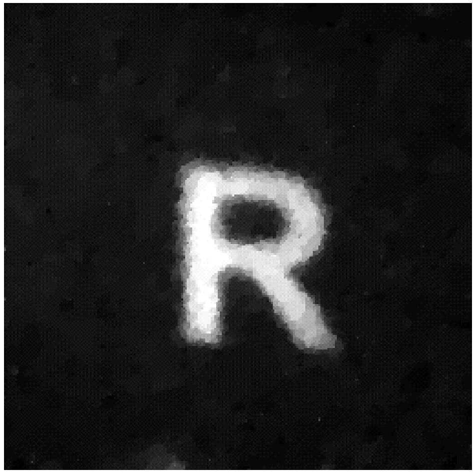
16384 Pixels
1600 Measurements
(10%)



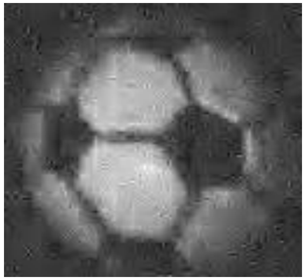
16384 Pixels
3300 Measurements
(20%)



65536 Pixels
1300 Measurements
(2%)

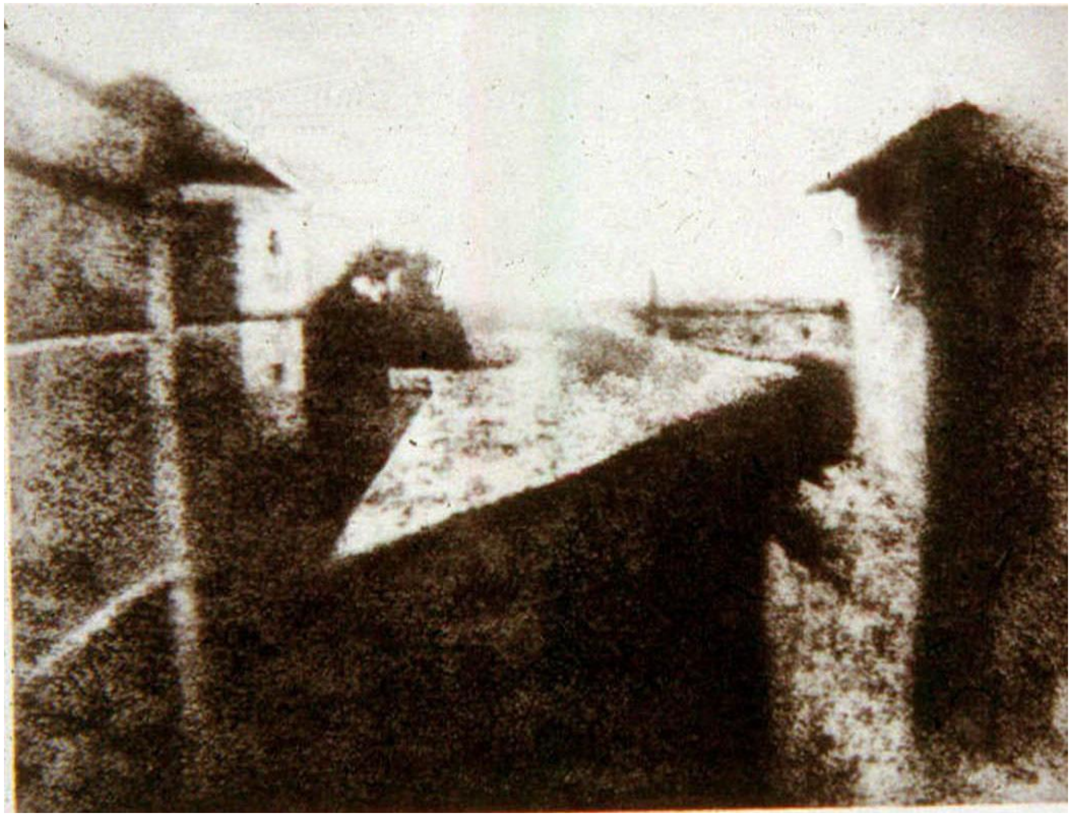


65536 Pixels
3300 Measurements
(5%)

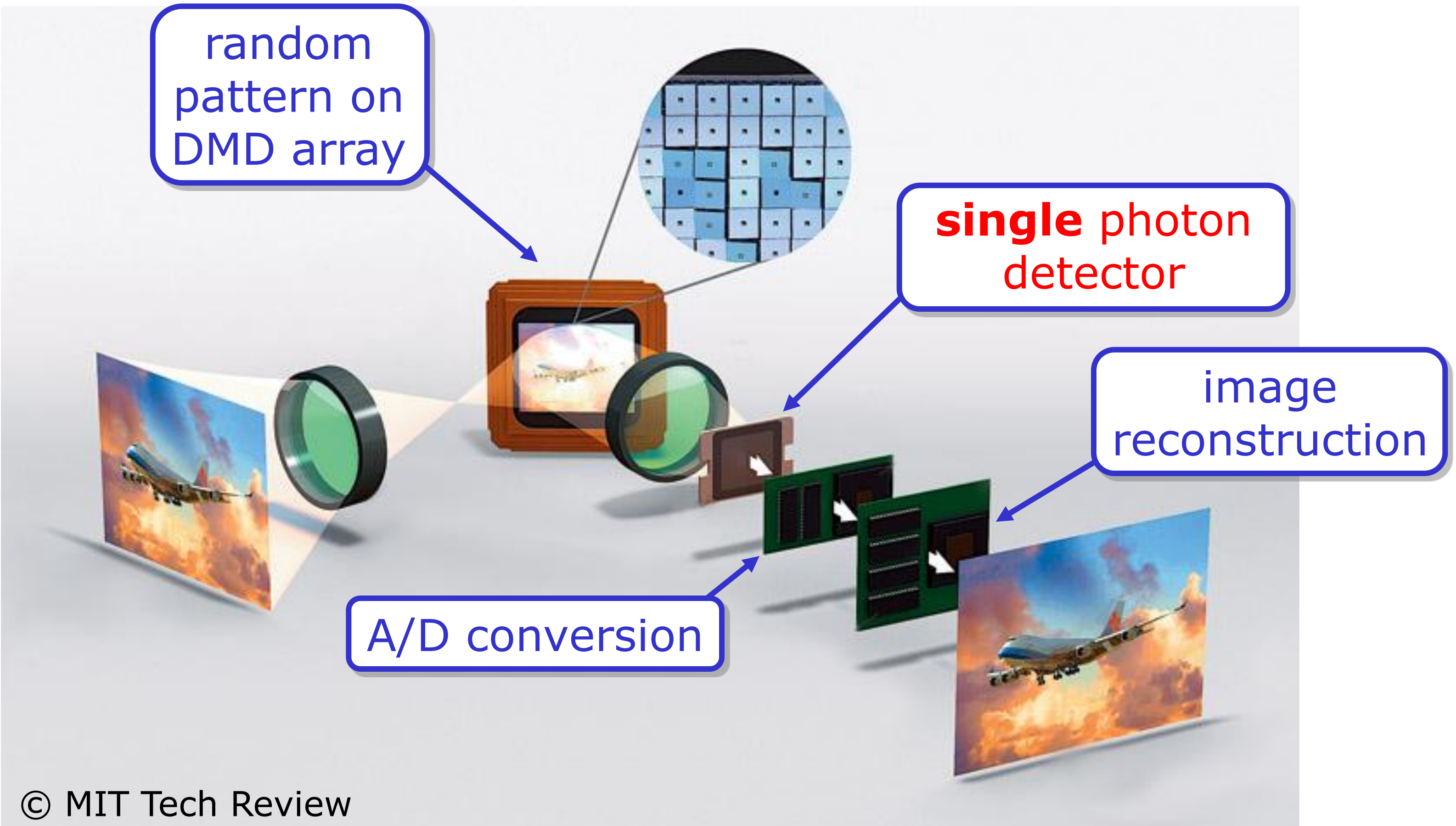


World's First Photograph

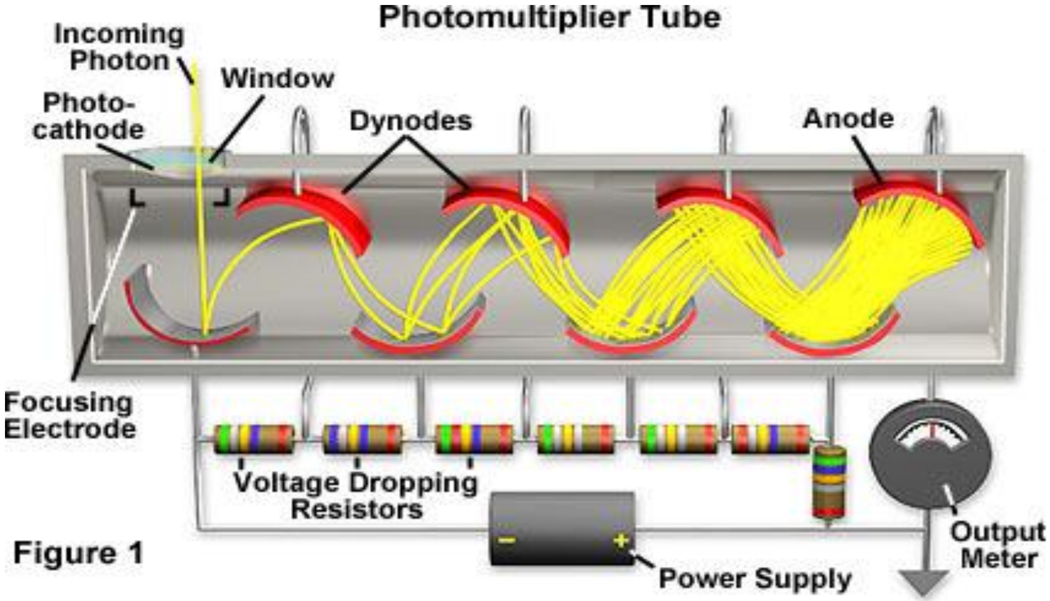
- 1826, Joseph Niepce
- Farm buildings and sky
- 8 hour exposure

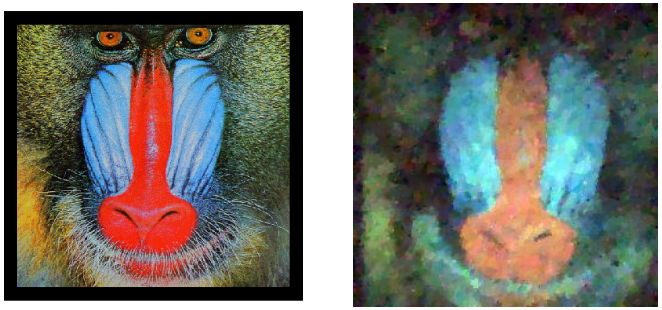


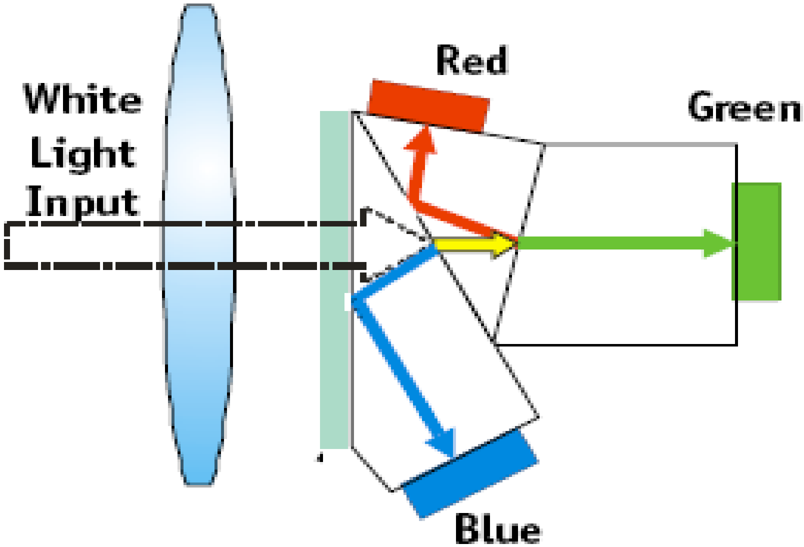
"Single-Pixel" Camera



Low-Light Imaging with PMT







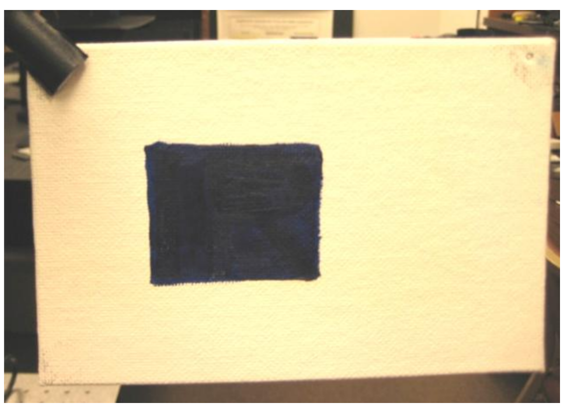
True color low-light imaging:

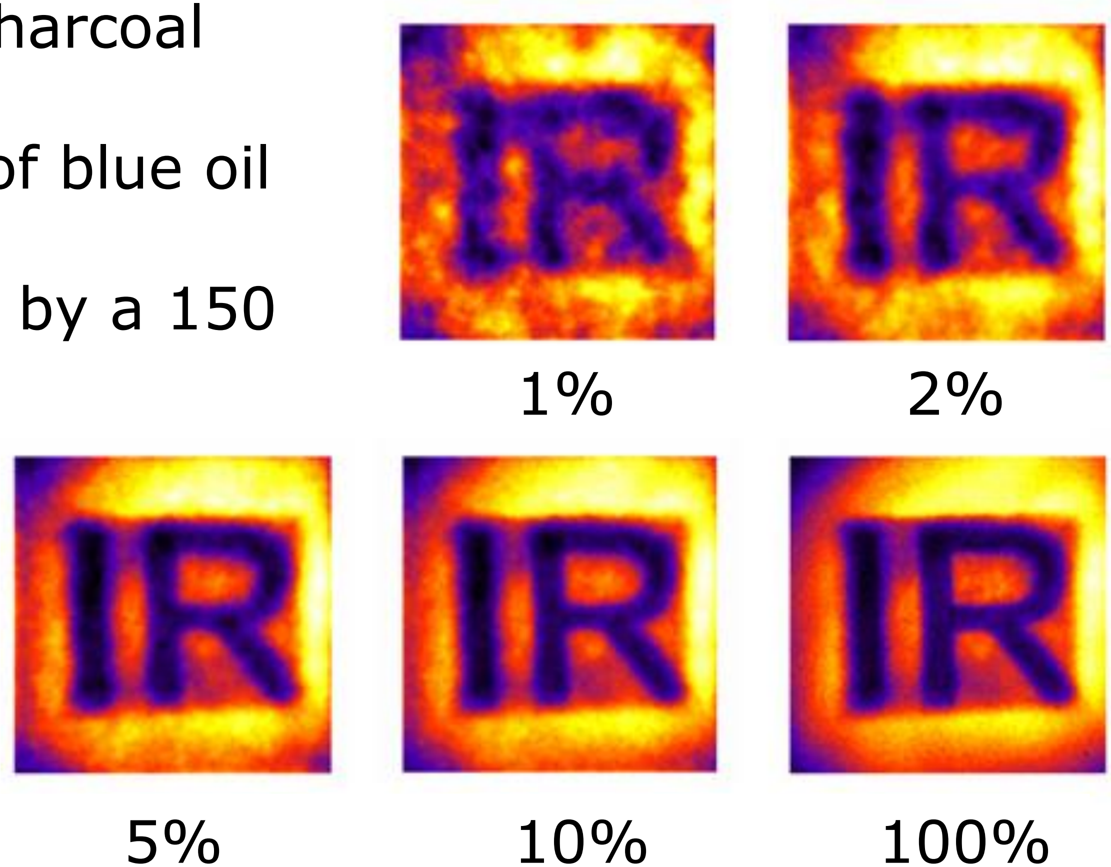
256 x 256 image with 10:1 compression

IR Imaging

Canvas board:

- "IR" written using charcoal pencil
- covered by a layer of blue oil paint
- scene is illuminated by a 150 watt halogen lamp

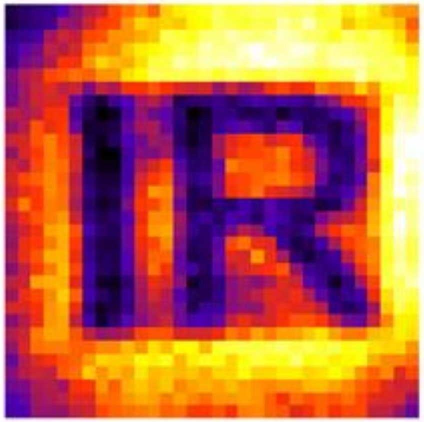




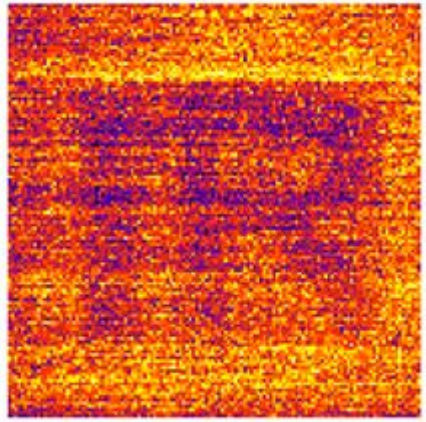
Reconstruction of 256 × 256 pixel image

IR Imaging

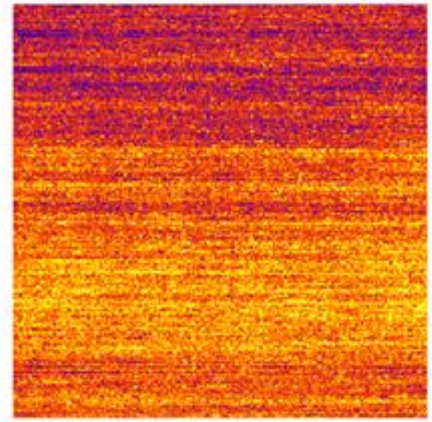
Raster scans: Light from only one pixel



32×32



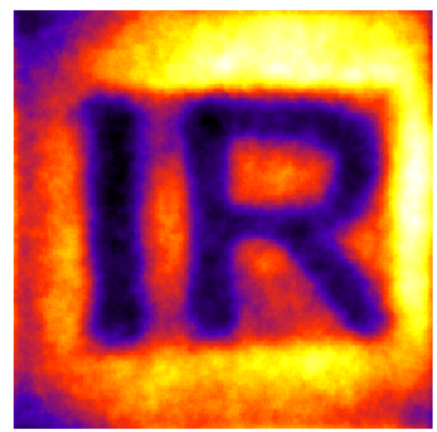
128×128



256×256

Compressive sensing:

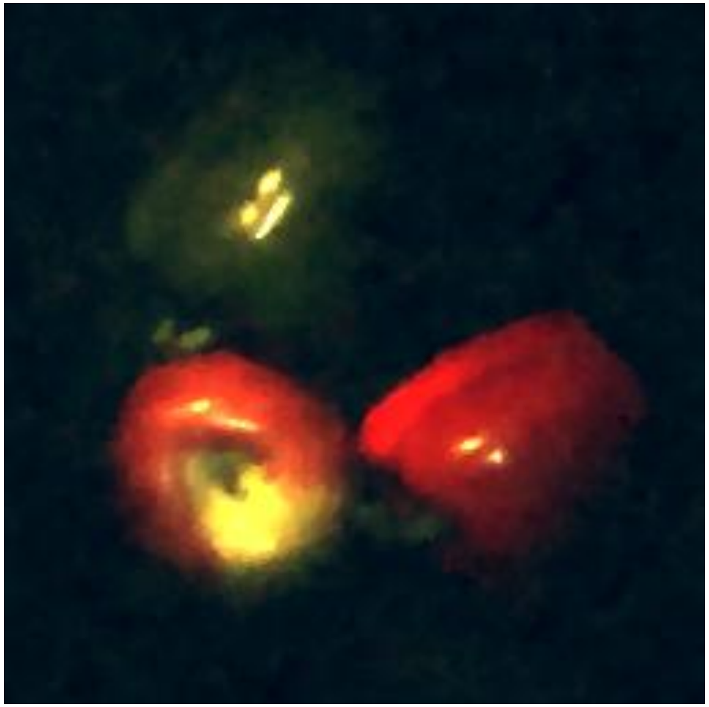
Light from half the pixels



256×256

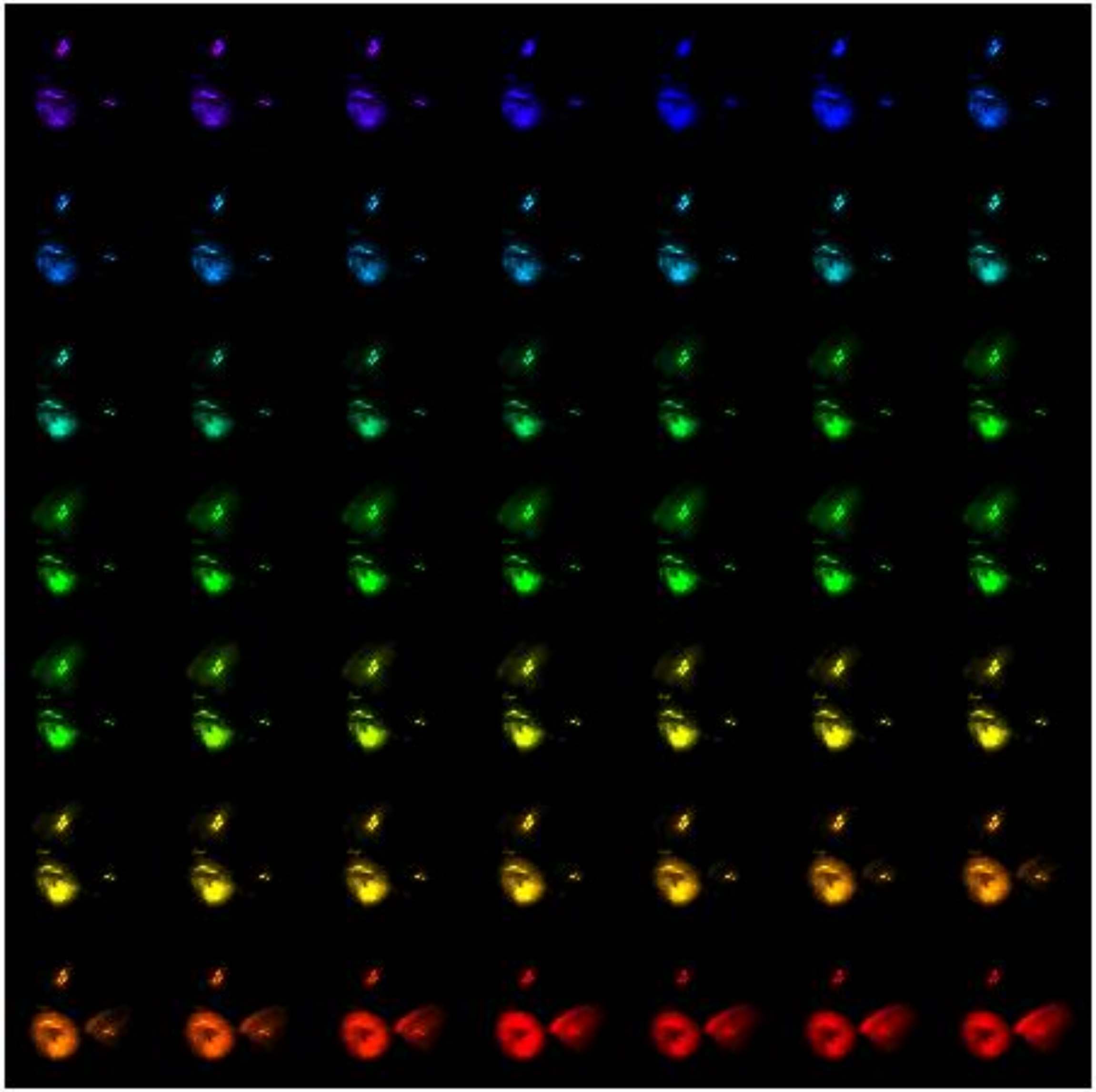
Hyperspectral Imaging

Sum of all bands

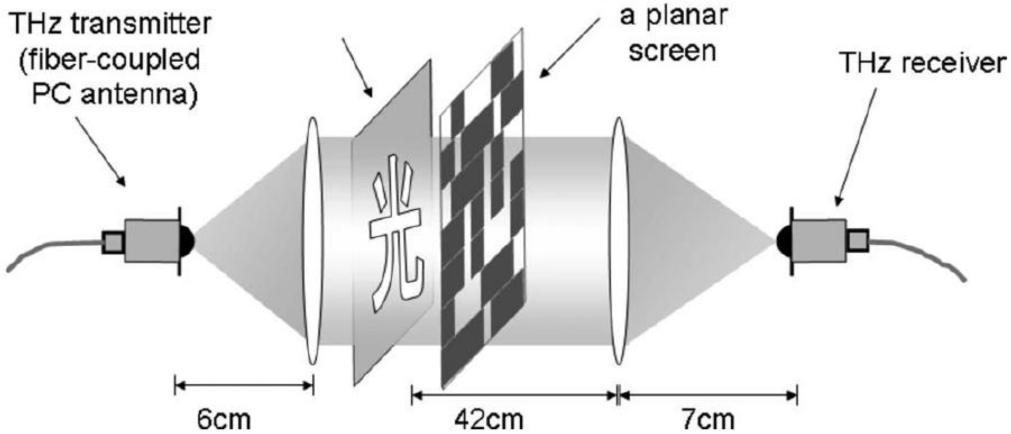


Real target

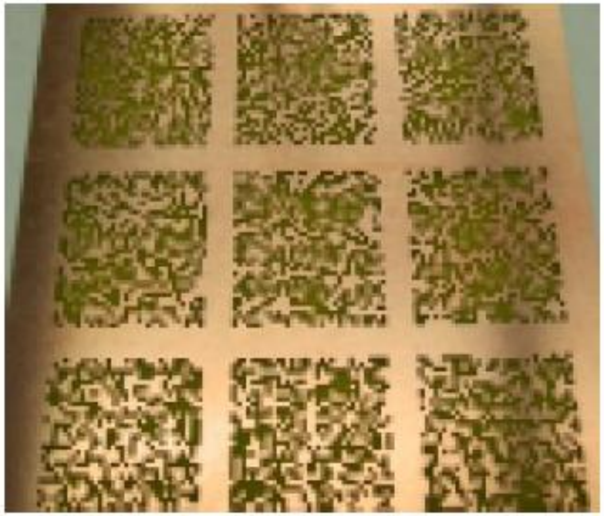
Hyperspectral Imaging

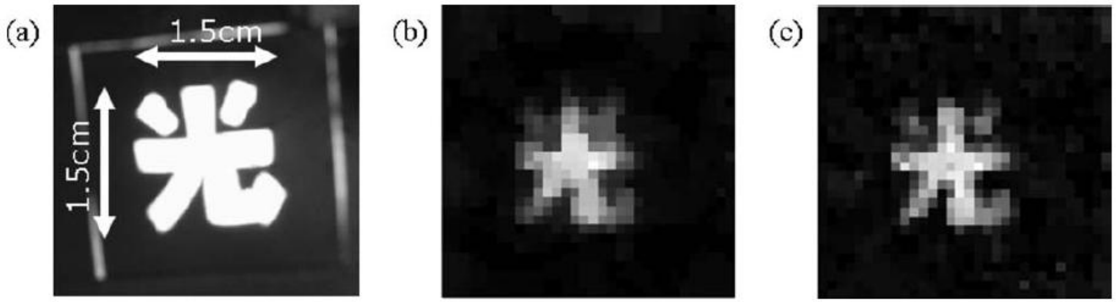


THZ Imaging



32 x 32 PCB masks

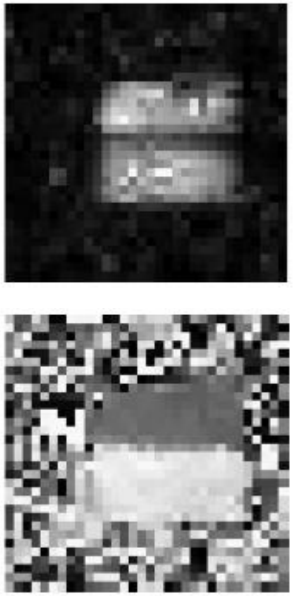




Object mask **300 measurements** **600 measurements**

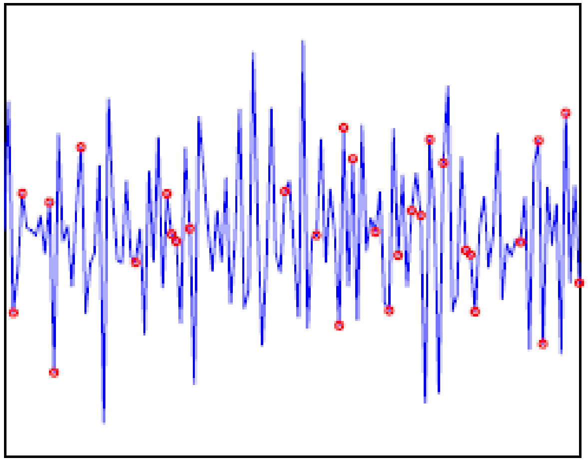
THz Amplitude

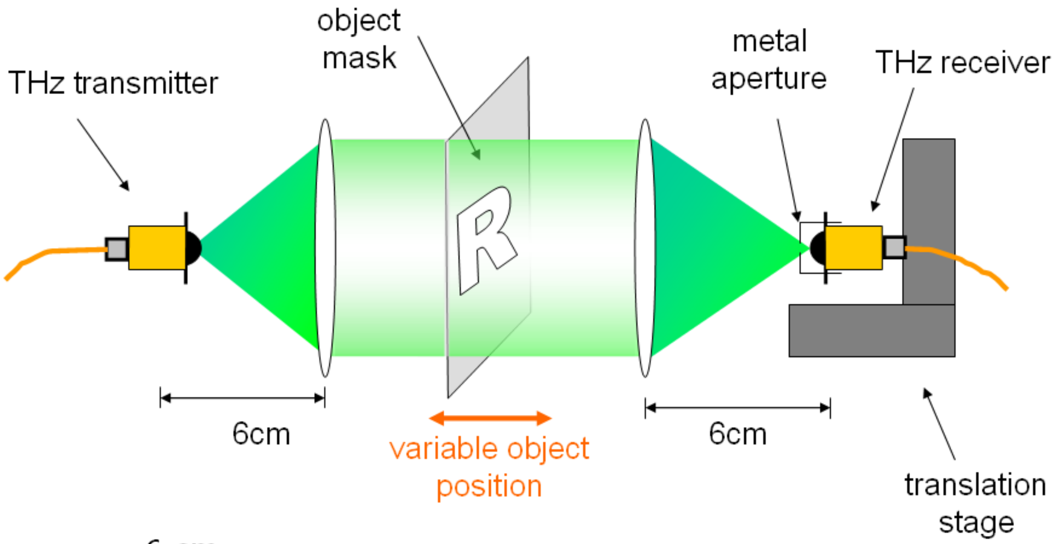
THz Phase

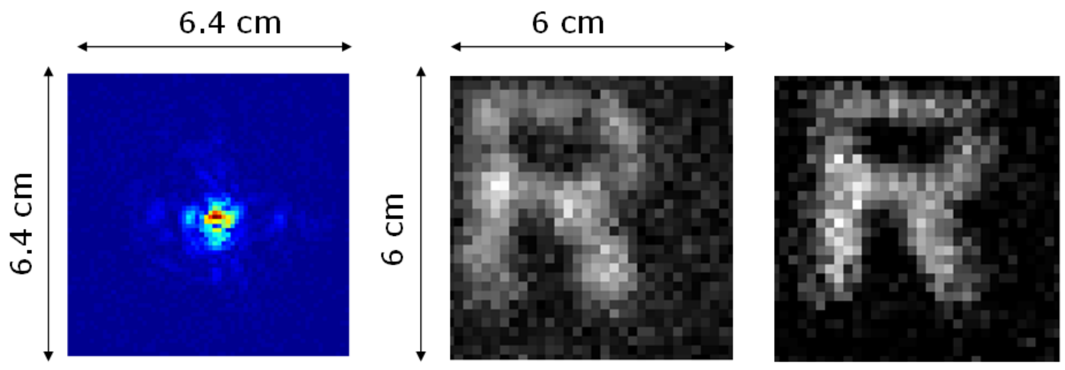


Mittleman Group, Rice University

THz Imaging 2: Sampling in Fourier







Fourier Transform
of object
(Magnitude-only)

CPR Reconstruction
(4096 measurements)

CSPR Reconstruction
(1000 measurements)

Mittleman Group, Rice University

Conclusions

- **Compressive sensing**
  - exploits signal sparsity/compressibility
  - integrates sensing with compression
  - enables new kinds of imaging/sensing devices

- Near/Medium-term applications
  - tomography/medical imaging
  - cameras and imagers where CCDs and CMOS arrays are blind
  - potential strategy to boost time-resolution in many imaging settings
  - ***electron microscopy?***

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