Compressive Sensing: Theory and Practice

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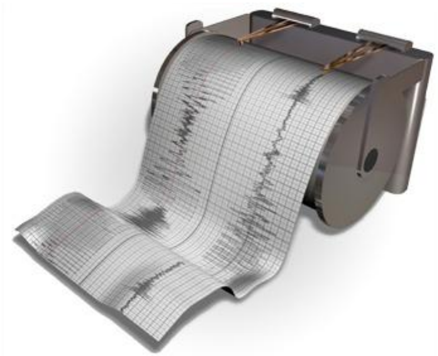
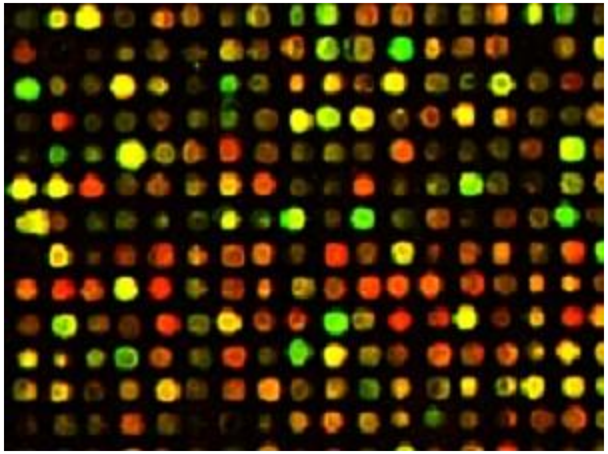
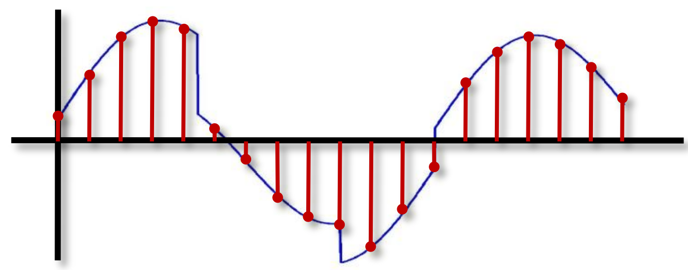
Sensor Explosion

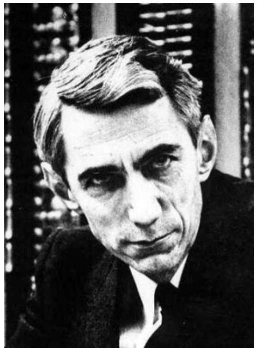



Digital Revolution

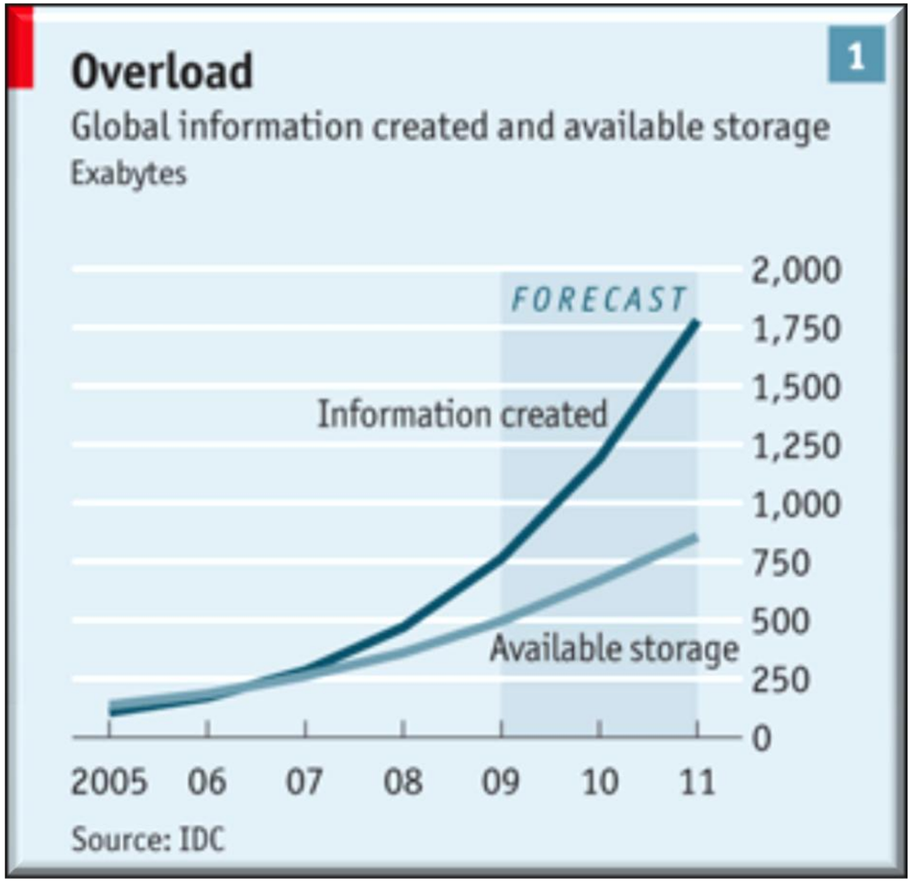


"If we sample a signal at twice its highest frequency, then we can recover it exactly."

Whittaker-Nyquist-Kotelnikov-Shannon



Data Deluge

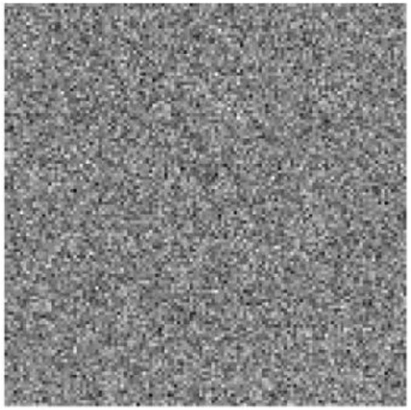


By 2011, ½ of digital universe will have no home

[The Economist – March 2010]

Dimensionality Reduction

Data is rarely intrinsically high-dimensional



Signals often obey ***low-dimensional models***

- sparsity
- manifolds
- low-rank matrices

The "intrinsic dimension" K can be much less than the "ambient dimension" N

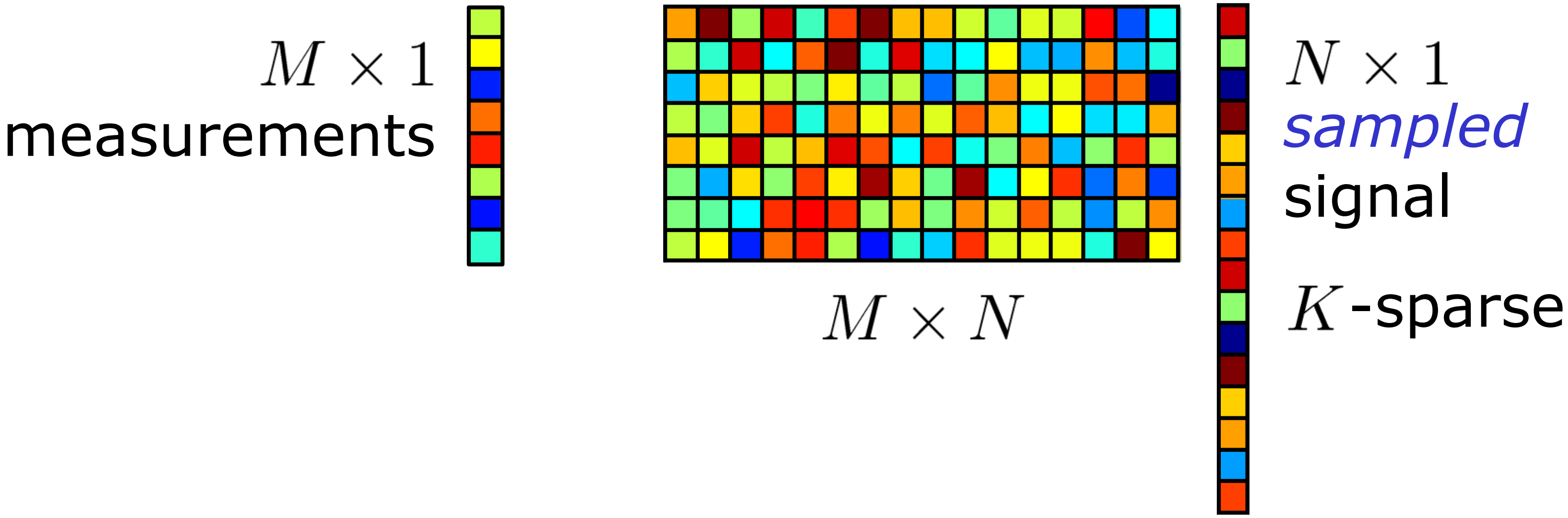
Compressive Sensing

Compressive Sensing

Compressive sensing [Donoho; Candes, Romberg, Tao – 2004]

Replace samples with general *linear measurements*

$$y = \Phi\, x$$



Sparsity Through History

William of Occam (1288-1348)

"Entities must not be multiplied unnecessarily"

"Simplicity is the ultimate sophistication"
-Leonardo da Vinci

"Make everything as simple as possible, but not simpler"
-Albert Einstein

Sparsity Through History

Gaspard Riche, baron de Prony (1795)

- algorithm for estimating the parameters of a few complex exponentials

$$x(t) = \sum_{i=1}^{K} \alpha_i e^{(j\omega_i + \lambda_i)t}$$

Sparsity Through History

Constantin Carathéodory (1907)

– given a sum of K sinusoids

$$x(t) = \sum_{i=1}^{K} \alpha_i e^{j\omega_i t}$$

we can recover $x(t)$ from $2K+1$ samples at *any* points in time

Sparsity Through History

Arne Beurling (1938)

– given a sum of K impulses

$$x(t) = \sum_{i=1}^{K} \alpha_i \delta(t - t_i)$$

we can recover $x(t)$ from only a piece of its Fourier transform

Sparsity Through History

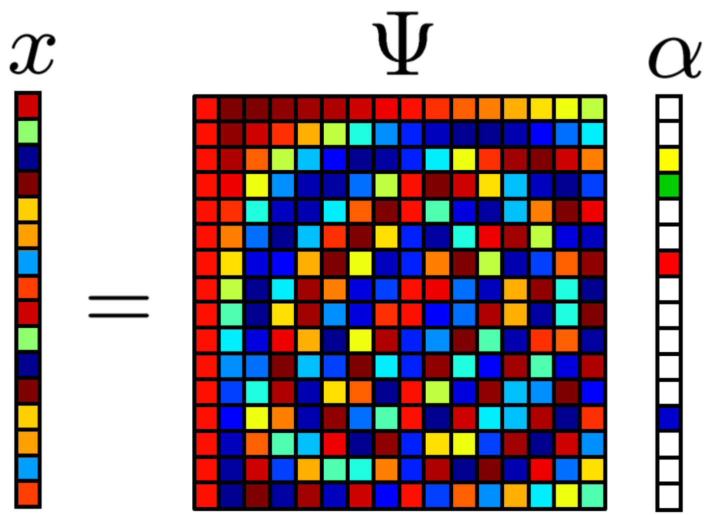
Ben "Tex" Logan (1965)

- given a signal $x(t)$ with bandlimit Ω, we can arbitrarily corrupt an interval of length $2\pi/\Omega$ and still be able to recover $x(t)$ no matter how it was corrupted

Sparsity

$$x = \sum_{j=1}^{N} \alpha_j \psi_j$$

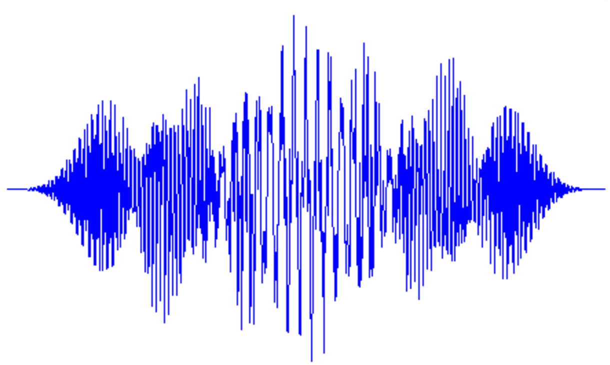
$$= \Psi \alpha$$

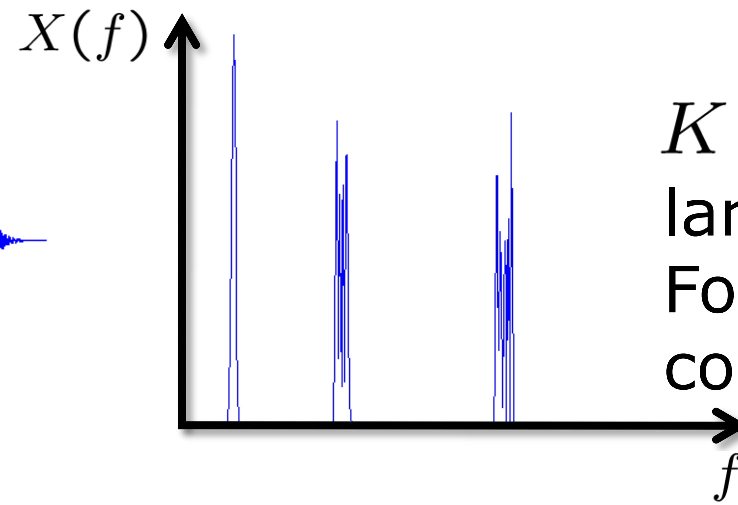


K nonzero entries

$\|\alpha\|_0 = K$

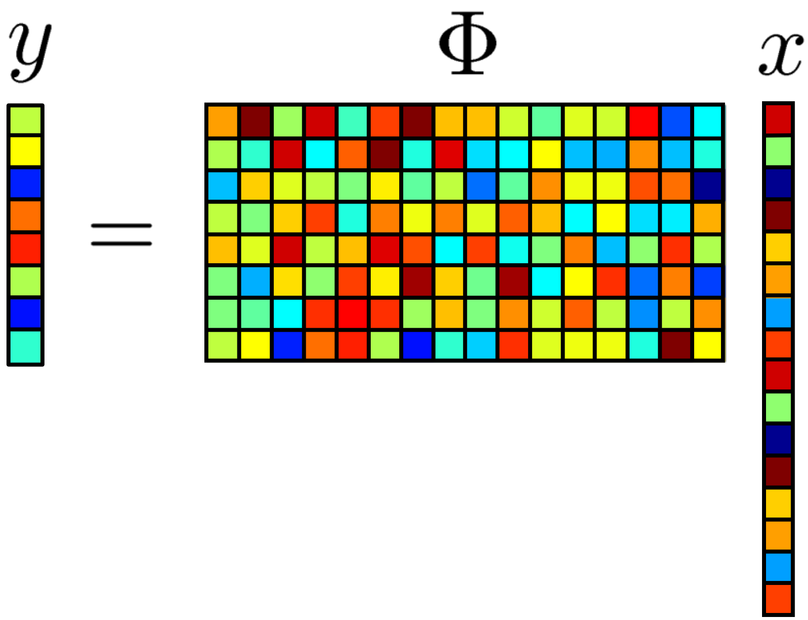
N samples





$K \ll N$ large Fourier coefficients

How Can We Exploit Sparsity



Two key theoretical questions:

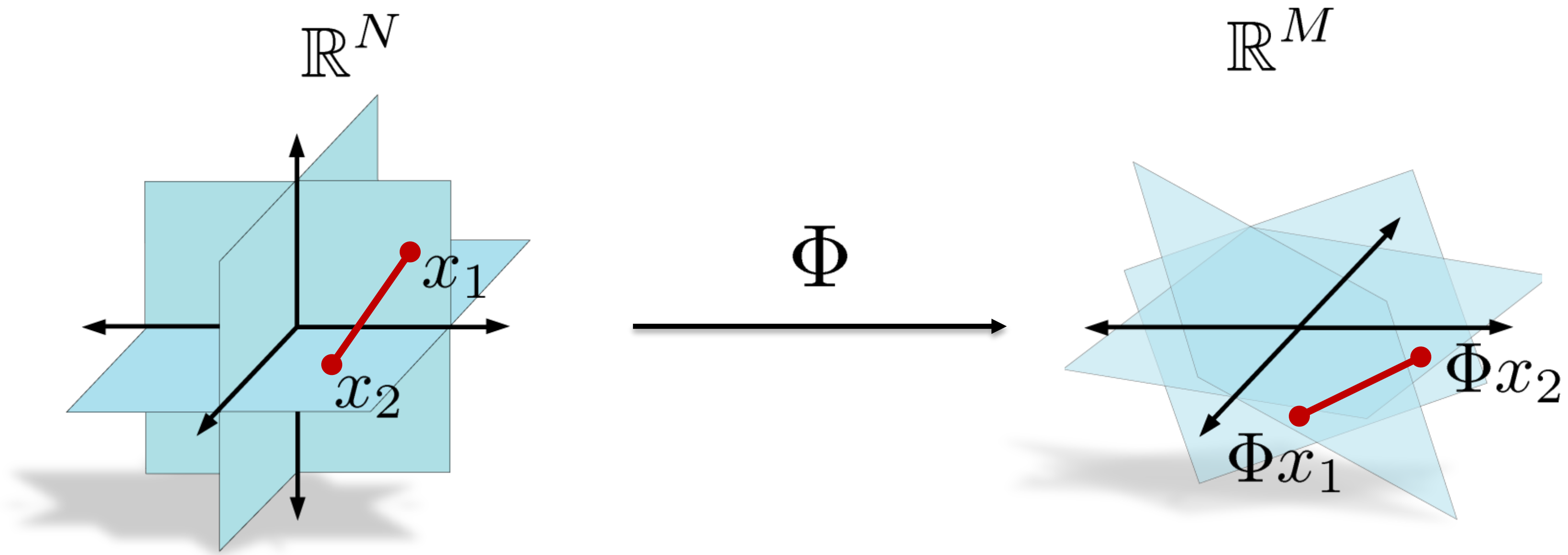
- How to design Φ that preserves the structure of x ?
- Algorithmically, how to recover x from the measurements y ?

Sensing Matrix Design

Restricted Isometry Property (RIP)

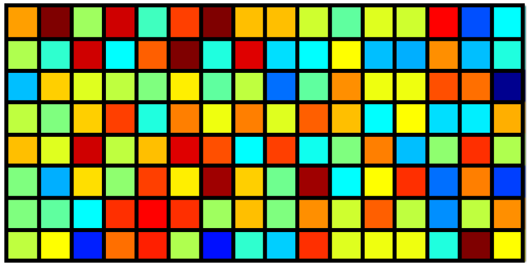
For any pair of K-sparse signals x_1 and x_2,

$$1 - \delta \leq \frac{\|\Phi x_1 - \Phi x_2\|_2^2}{\|x_1 - x_2\|_2^2} \leq 1 + \delta.$$



RIP Matrix: Option 1

- Choose a ***random matrix***
  - fill out the entries of Φ with i.i.d. samples from a sub-Gaussian distribution
  - project onto a "random subspace"



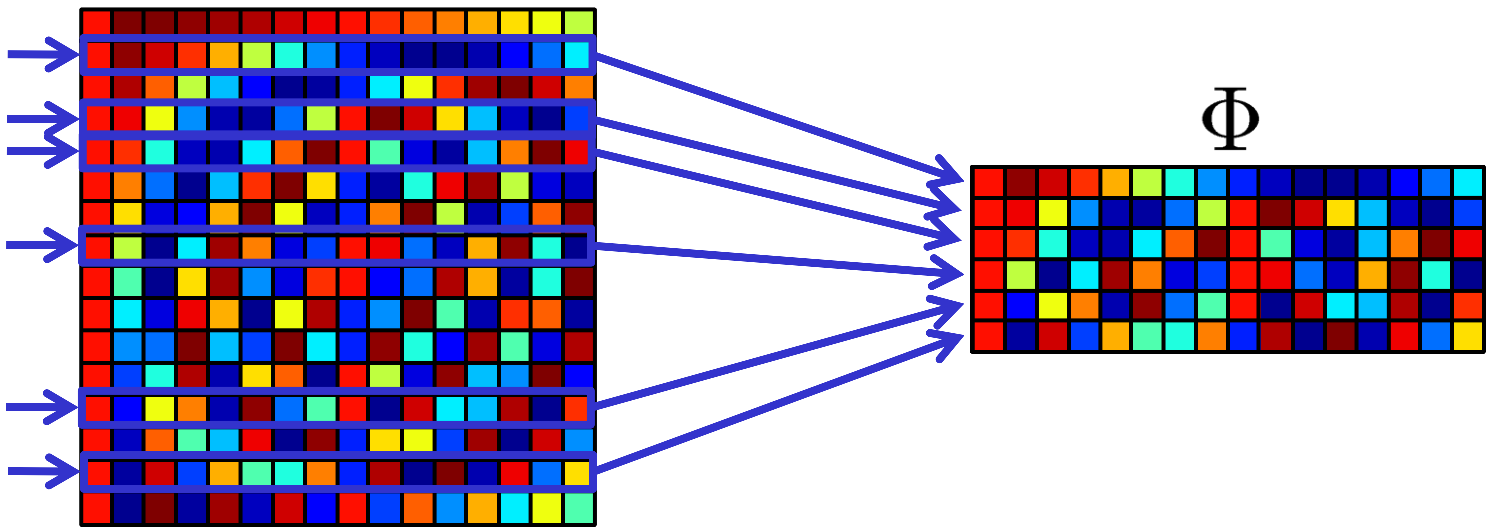
- Deep connections with random matrix theory and Johnson-Lindenstrauss Lemma

$$M = O(K \log(N/K)) \ll N$$

[Baraniuk, M.D., DeVore, Wakin – Const. Approx. 2008]

RIP Matrix: Option 2

- Random Fourier submatrix



$$M = O(K \log^p (N/K)) \ll N$$

[Candes and Tao – Trans. Information Theory 2006]

Hallmarks of Random Measurements

Stable

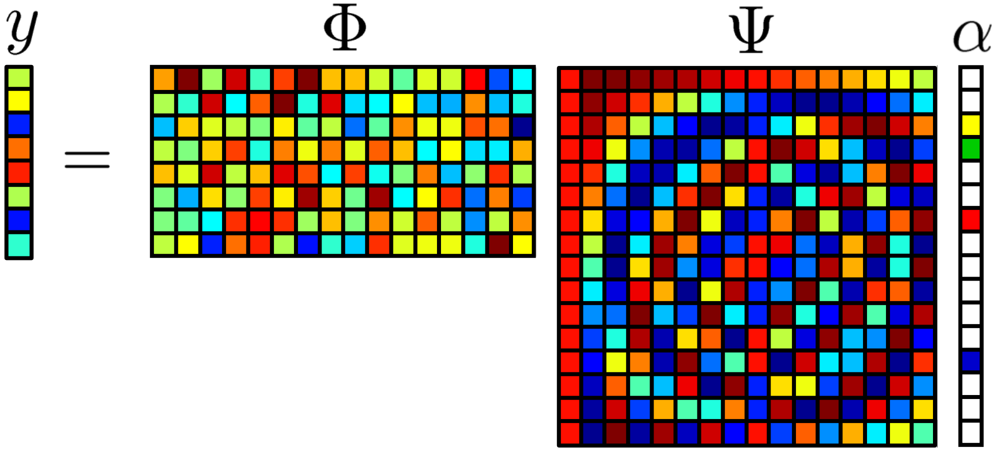
Φ will preserve information, be robust to noise

Democratic

Each measurement has "equal weight"

Universal

Random Φ will work with ***any*** fixed orthonormal basis



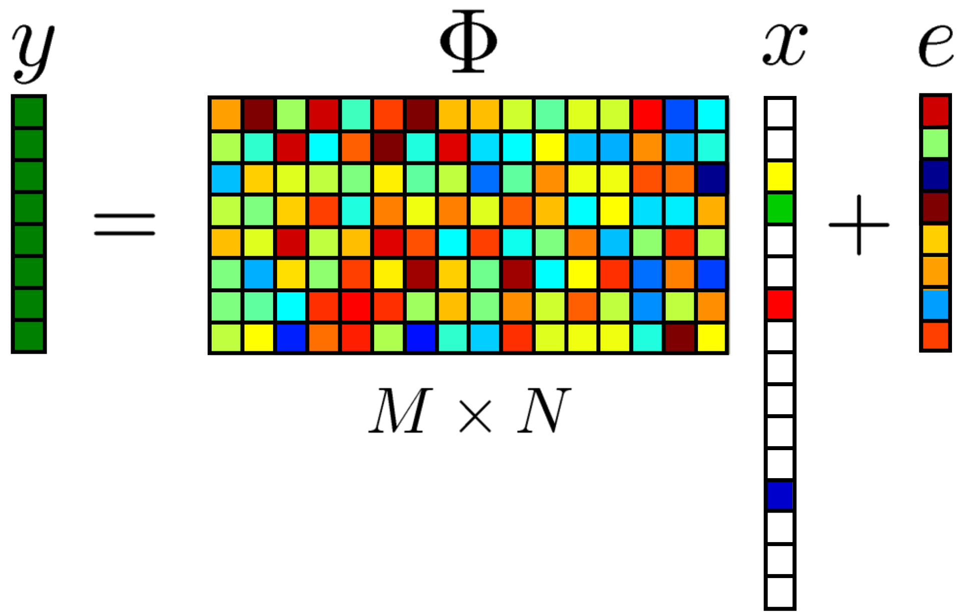
Signal Recovery

Signal Recovery

given $y = \Phi x + e$

find x

ill-posed inverse problem



Sparse Recovery: Noiseless Case

given $y = \Phi x$
find x

ℓ_0-minimization:
$$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \|x\|_0 \quad \text{s.t.} \quad y = \Phi x$$
nonconvex
NP-Hard

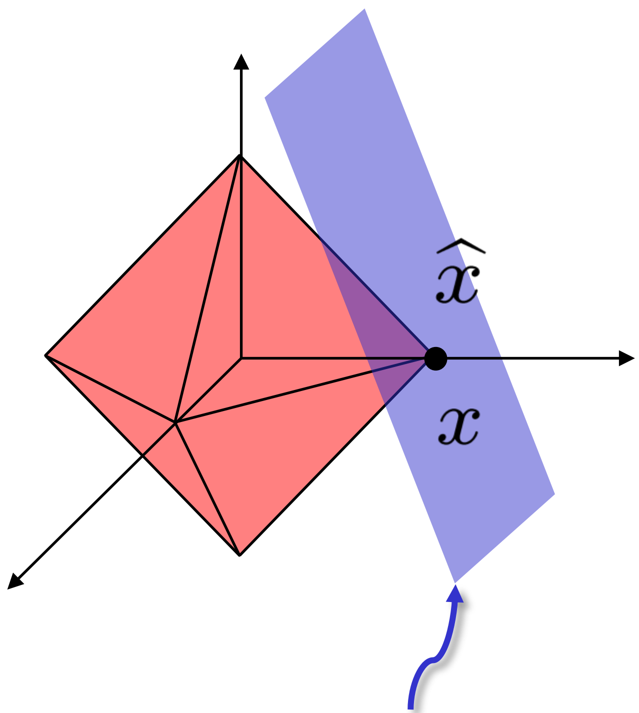
ℓ_1-minimization:
$$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \|x\|_1 \quad \text{s.t.} \quad y = \Phi x$$
linear program

If Φ satisfies the RIP, then ℓ_0 and ℓ_1 are equivalent!

Why ℓ_1-Minimization Works

$$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \|x\|_1$$

$$\text{s.t.} \quad y = \Phi x$$



$$\{x' : \Phi x' = y\}$$

Recovery in Noise

- Optimization-based methods

$$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \|x\|_1 \quad \text{s.t.} \quad \|y - \Phi x\|_2 \leq \epsilon$$

- Greedy/Iterative algorithms
  - OMP, StOMP, ROMP, CoSaMP, Thresh, SP, IHT

$$\boxed{\|\widehat{x} - x\|_2 \leq C_0 \|e\|_2 + C_1 \frac{\|x - x_K\|_1}{\sqrt{K}}}$$

Compressive Sensing in Practice

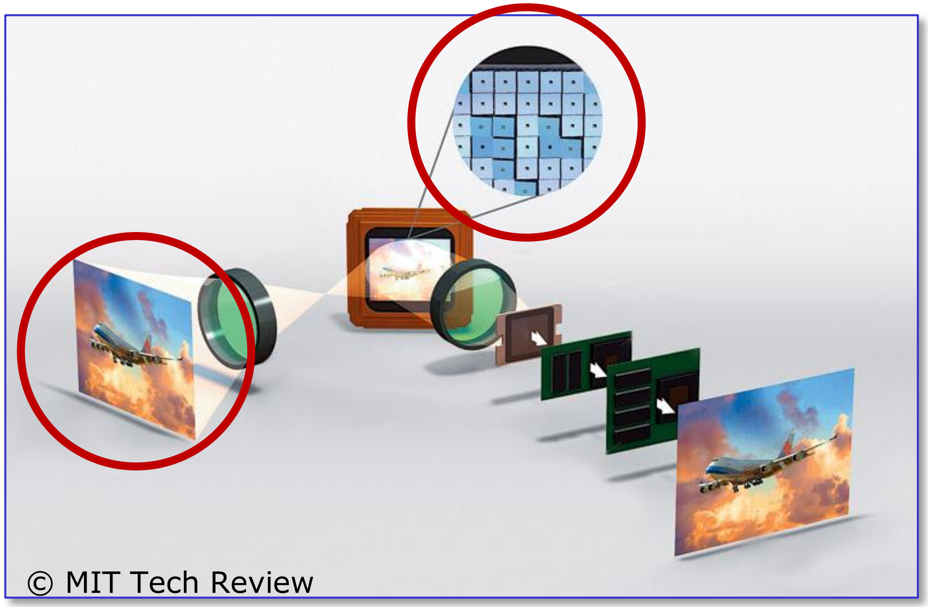
Compressive Sensing in Practice

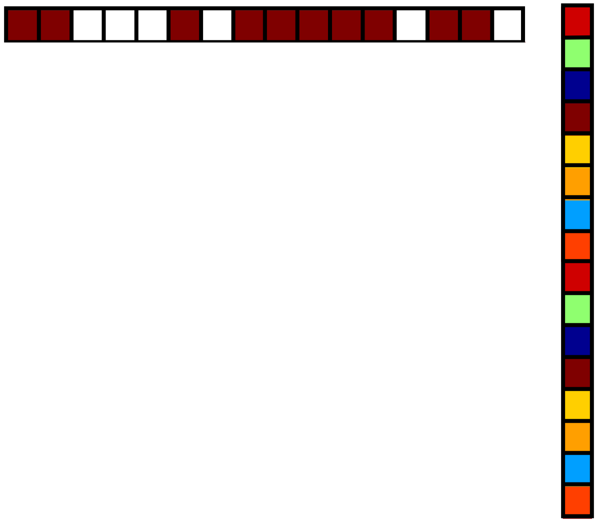
- Tomography in medical imaging
  - each projection gives you a set of Fourier coefficients
  - fewer measurements mean
    - more patients
    - sharper images
    - less radiation exposure

- Wideband signal acquisition
  - framework for acquiring sparse, wideband signals
  - ideal for some surveillance applications

- "Single-pixel" camera

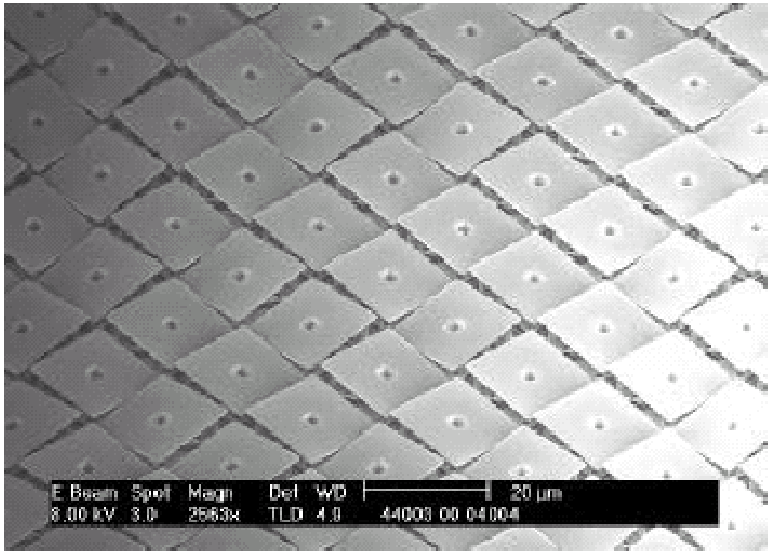
“Single-Pixel” Camera





[Duarte, M.D., Takhar, Laska, Sun, Kelly, Baraniuk – Sig. Proc. Mag. 2008]

TI Digital Micromirror Device



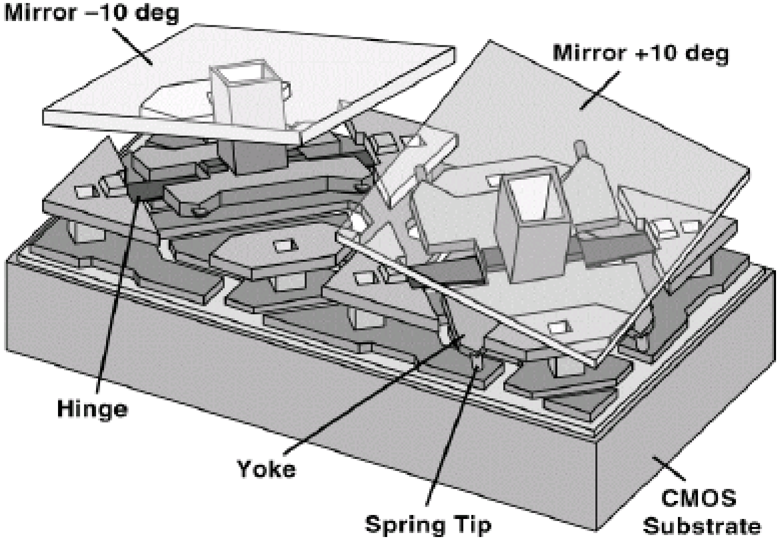
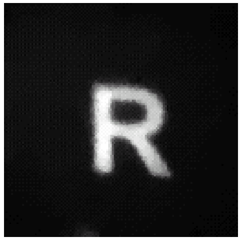
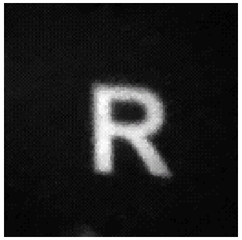


Image Acquisition

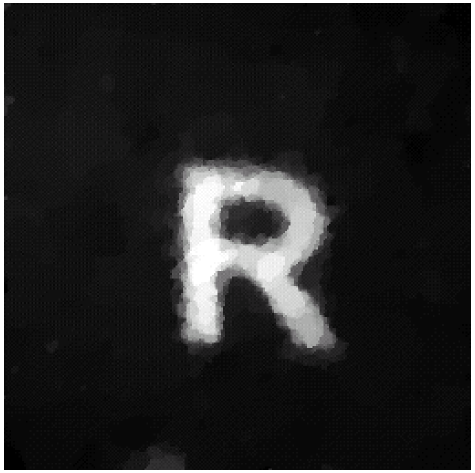
Original



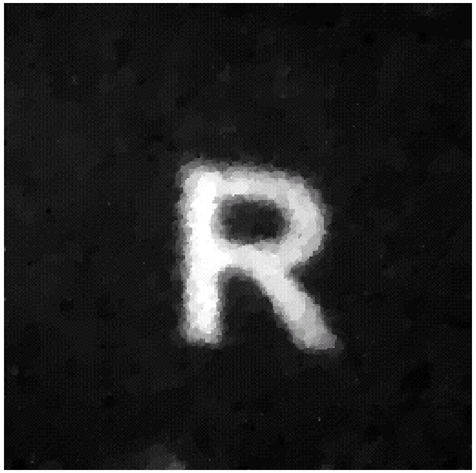
16384 Pixels
1600 Measurements
(10%)



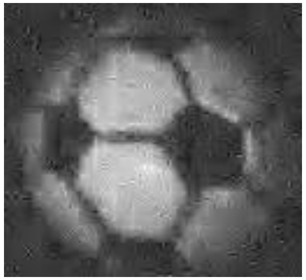
16384 Pixels
3300 Measurements
(20%)



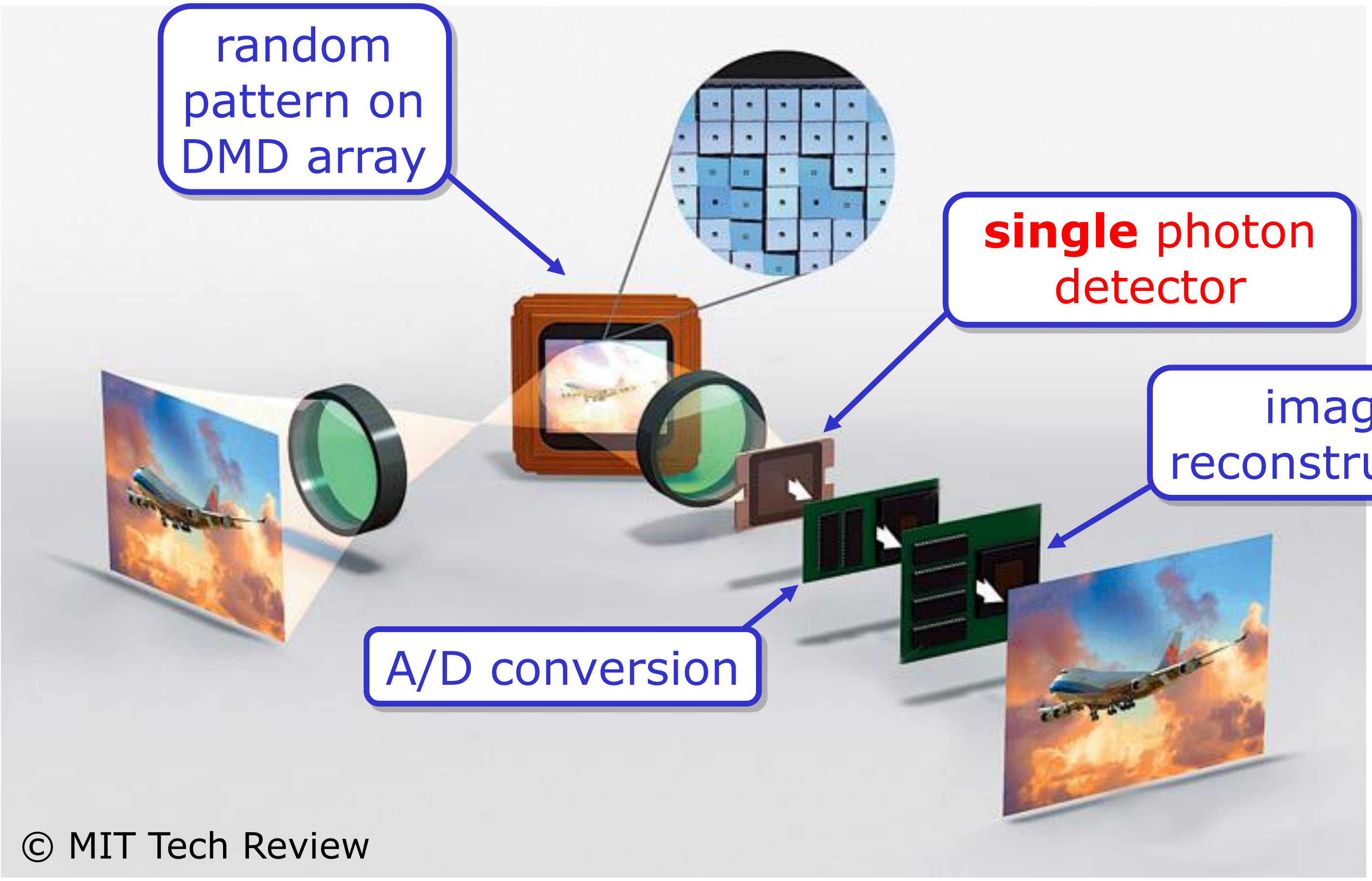
65536 Pixels
1300 Measurements
(2%)



65536 Pixels
3300 Measurements
(5%)



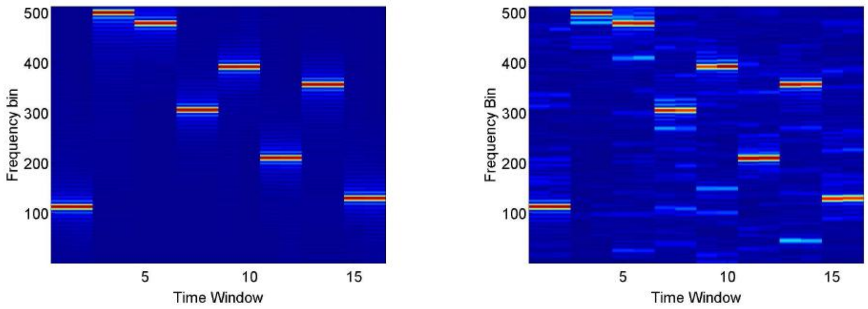
"Single-Pixel" Camera



Conclusions

- **Compressive sensing**
  - exploits signal sparsity/compressibility
  - integrates sensing with compression
  - enables new kinds of imaging/sensing devices

- Near/Medium-term applications
  - tomography/medical imaging
  - imagers where CCDs and CMOS arrays are blind
  - wideband A/D converters



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