Lost Without A Compass:

Nonmetric Triangulation and Landmark Multidimensional Scaling

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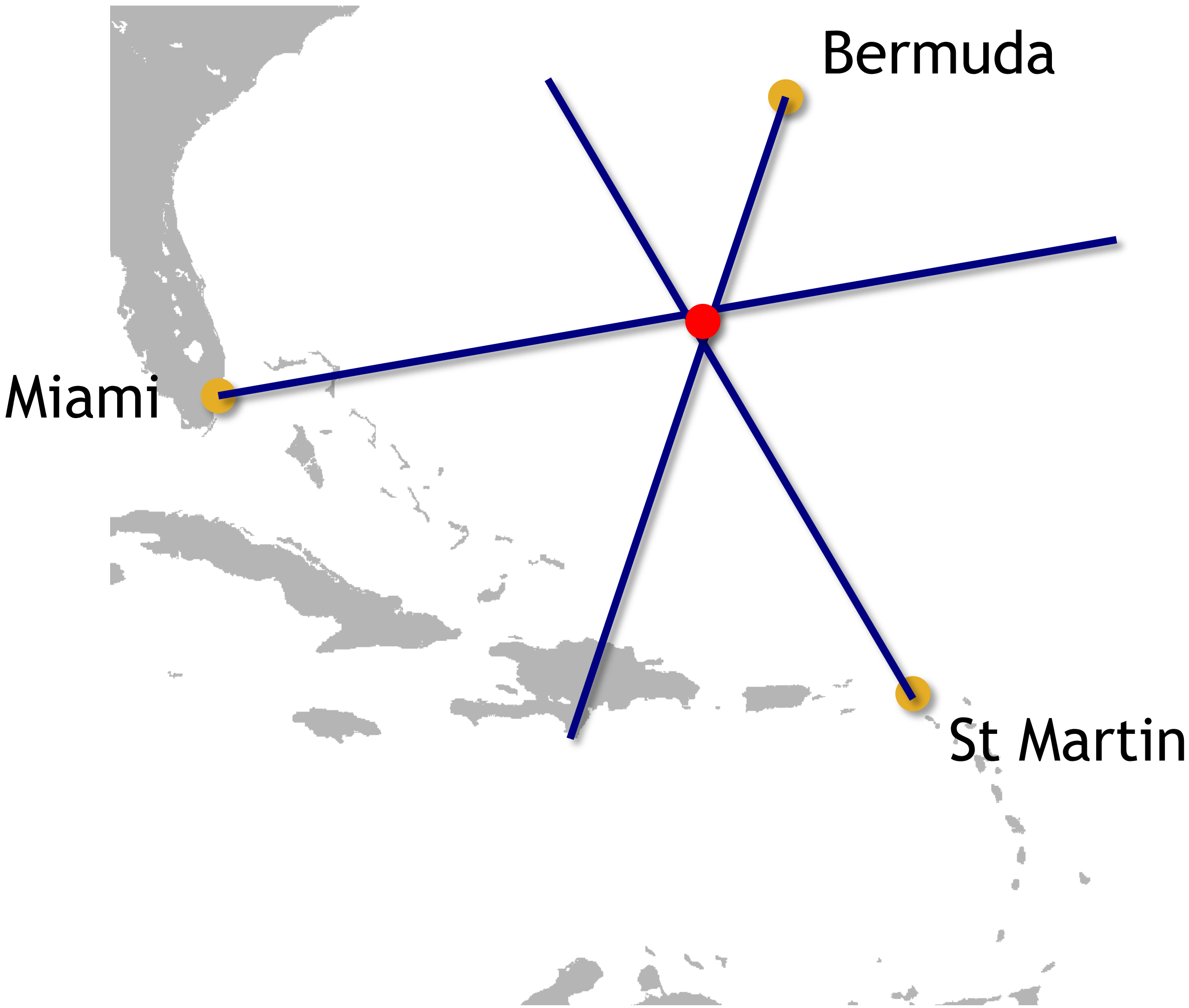
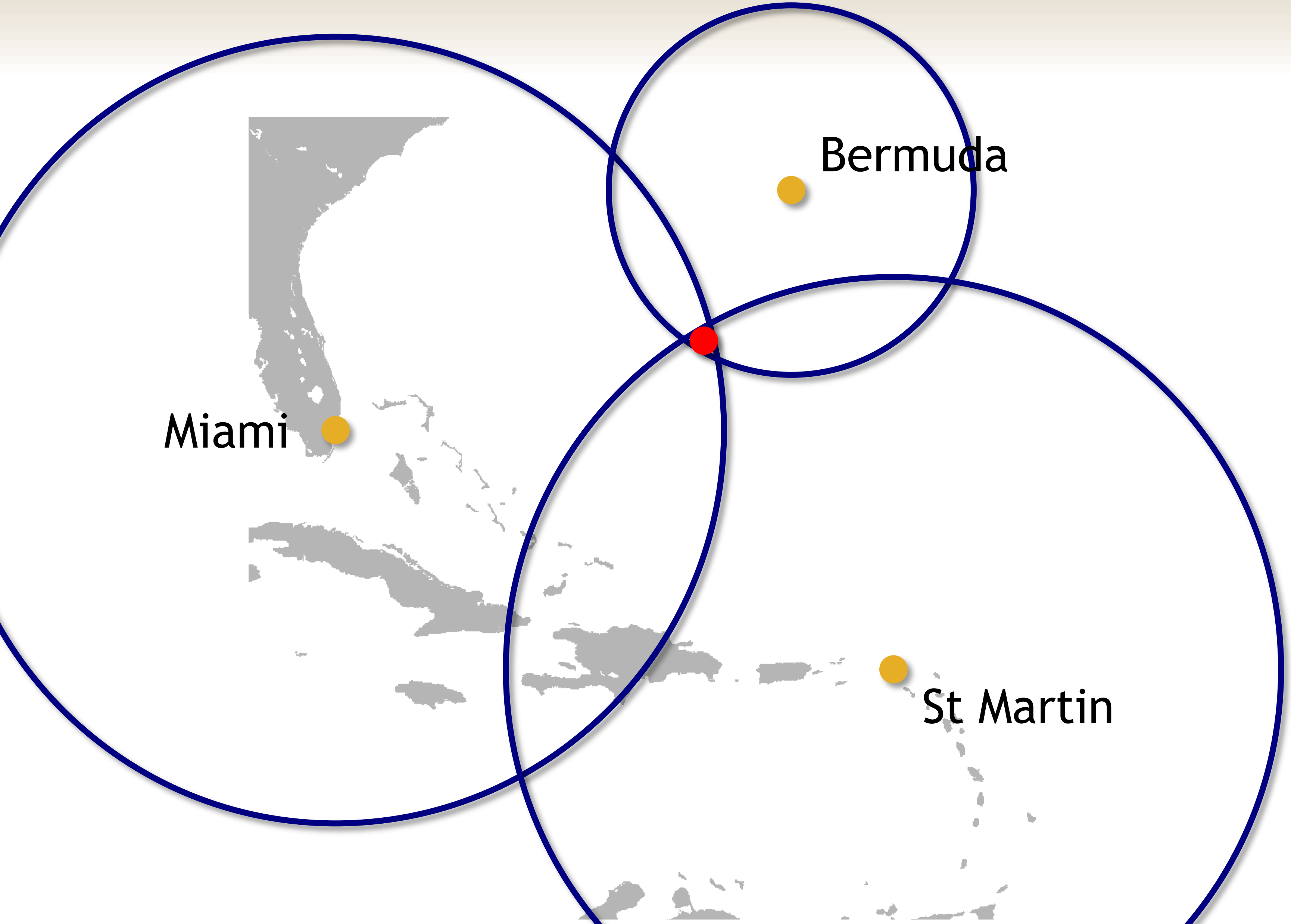
The triangulation problem



The triangulation problem



Least-squares triangulation

$x_1, \ldots, x_n \in \mathbb{R}^k$: "Landmark points" (locations known)

$x^* \in \mathbb{R}^k$: query point (location unknown)

Observe $d_j \approx \|x^* - x_j\|_2^2$ for $j = 1, \ldots, n$.

How to estimate $x^* \in \mathbb{R}^k$?

$$d_1 = \|x^* - x_1\|_2^2 = \|x^*\|_2^2 + \|x_1\|_2^2 - 2\langle x^*, x_1\rangle$$

$$d_j = \|x^* - x_j\|_2^2 = \|x^*\|_2^2 + \|x_j\|_2^2 - 2\langle x^*, x_j\rangle$$

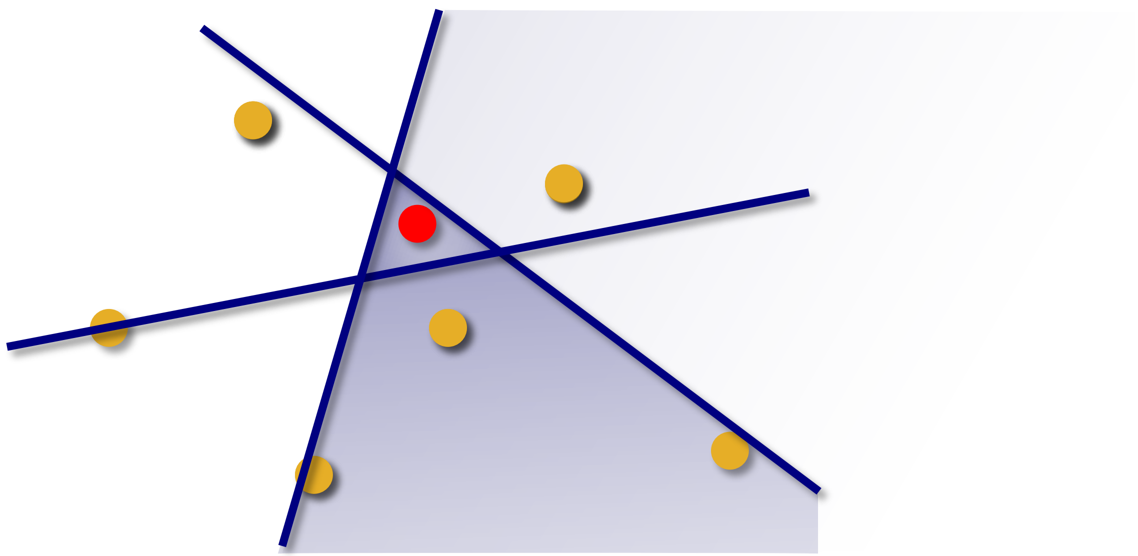
$$\langle x^*, x_j - x_1\rangle = \frac{\|x_j\|_2^2 - \|x_1\|_2^2 + d_1 - d_j}{2}$$

$n - 1$ linear equations, k unknowns

Nonmetric triangulation

What if we can't measure distances?

- closer to Bermuda than Miami
- closer to Miami than St Martin
- …



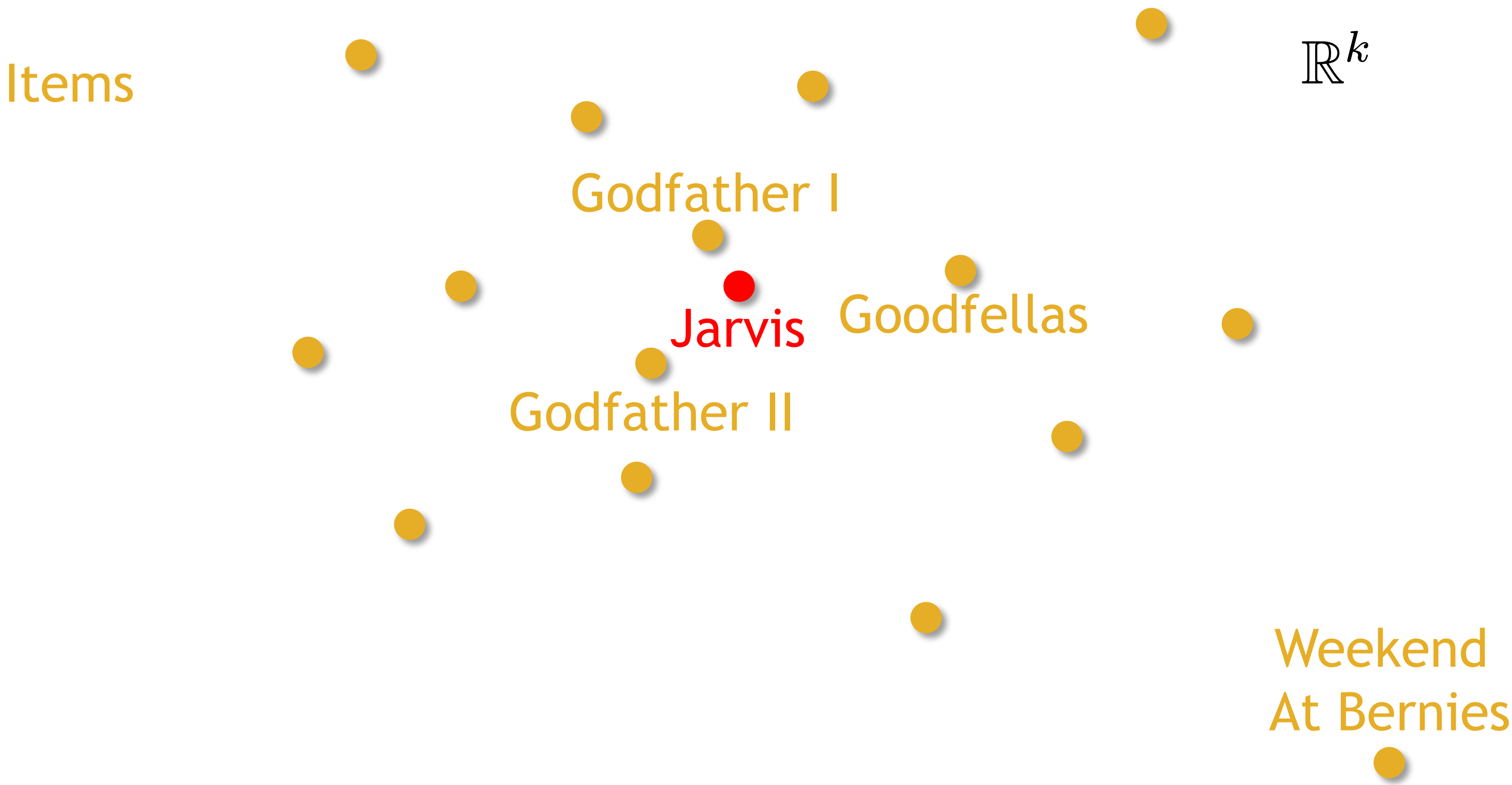
Observations: $\mathcal{T}$ such that for all $(i, j) \in \mathcal{T}$

$$\|x^* - x_i\|_2^2 < \|x^* - x_j\|_2^2$$

Applications

- Localization
- Nonmetric multidimensional scaling
  - useful tool for data exploration
  - observations are often nonmetric (e.g., pairwise comparisons)
  - adding new data to an existing embedding
  - scalable algorithms via “landmark points”

Ideal point model of preference



Convex optimization approach

Observations: $\mathcal{T}$ such that for all $(i,j) \in \mathcal{T}$

$$\|x^* - x_i\|_2^2 < \|x^* - x_j\|_2^2$$

Recall that if $d_i = \|x^* - x_i\|_2^2$, then

$$\langle x^*, x_i - x_j \rangle = \frac{\|x_i\|_2^2 - \|x_j\|_2^2 + d_j - d_i}{2}$$

Thus, if $\|x^* - x_i\|_2^2 < \|x^* - x_j\|_2^2$, then

$$\langle x^*, x_i - x_j \rangle > \frac{\|x_i\|_2^2 - \|x_j\|_2^2}{2}$$

Convex optimization approach

Use observations to define $|\mathcal{T}|$ constraints

$$\widehat{x} = \underset{x}{\text{argmin}} \, \|x\|_2^2$$

$$\text{s.t.} \quad \langle x, x_i - x_j \rangle > \frac{\|x_i\|_2^2 - \|x_j\|_2^2}{2} \quad (i,j) \in \mathcal{T}$$

Solution can be highly sensitive to noise

Robust version

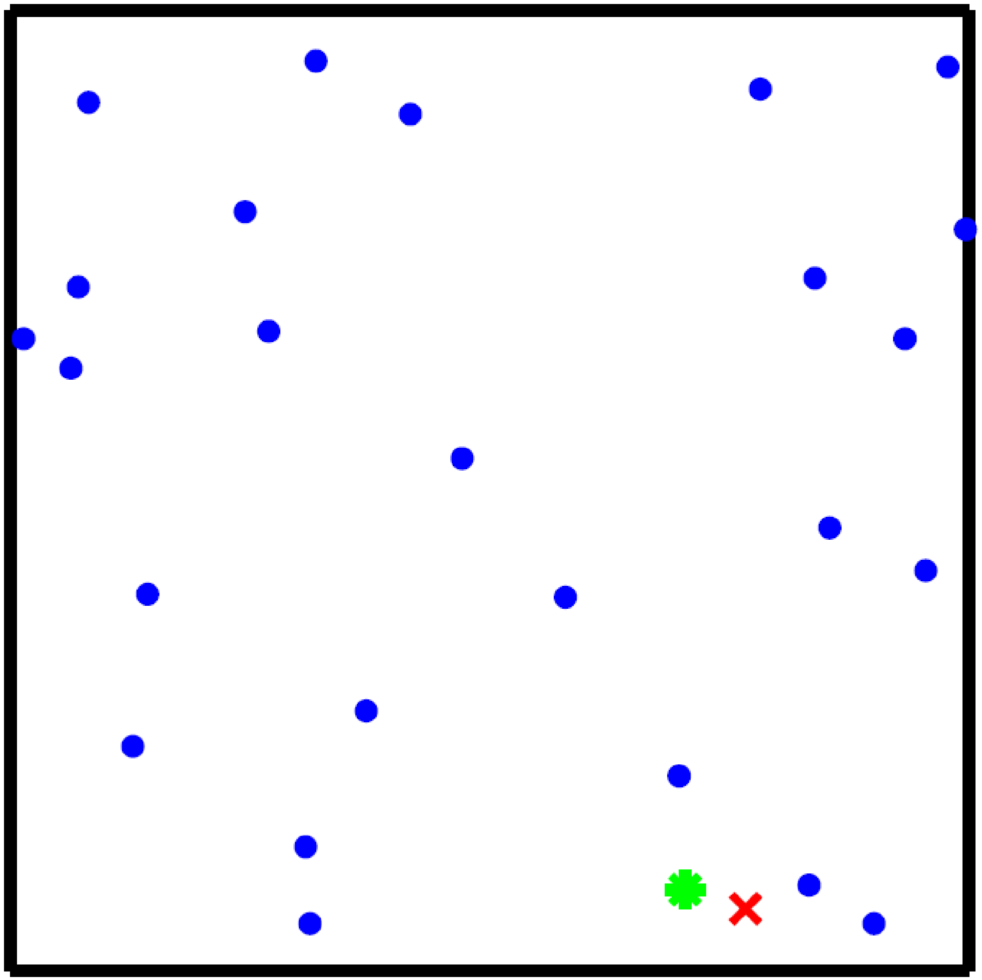
Introduce "slack variables" to allow some constraints to be violated

$$\widehat{x} = \operatorname*{argmin}_{x,\xi} \|x\|_2^2 + C \sum_{(i,j)\in\mathcal{T}} \xi_{i,j}$$

$$\text{s.t.} \quad \langle x, x_i - x_j \rangle \geq \frac{\|x_i\|_2^2 - \|x_j\|_2^2}{2} - \xi_{i,j}$$

$$\xi_{i,j} \geq 0 \qquad (i,j) \in \mathcal{T}$$

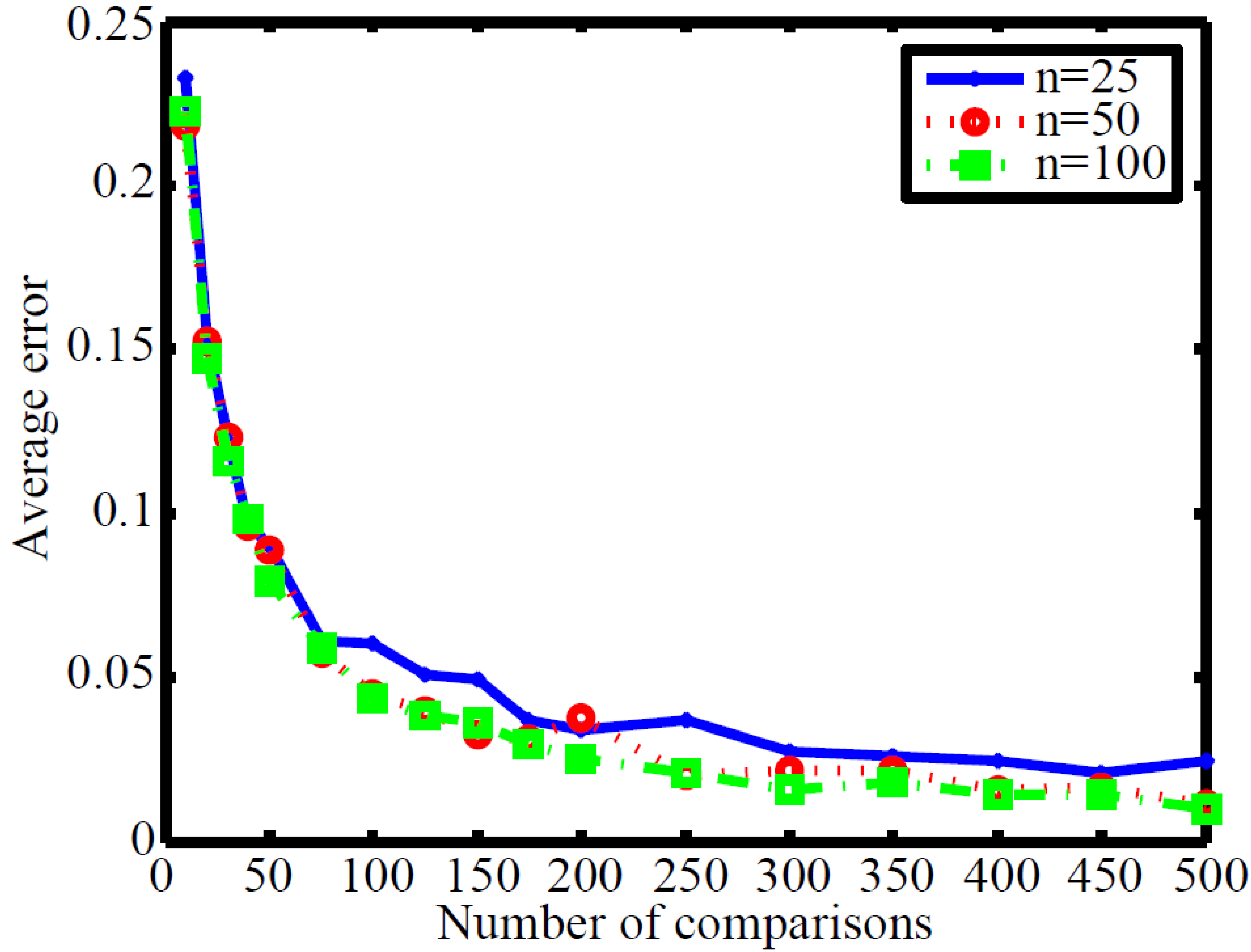
Example



$$n = 25$$

$$|\mathcal{T}| = 50$$

Localization error



Nonmetric MDS

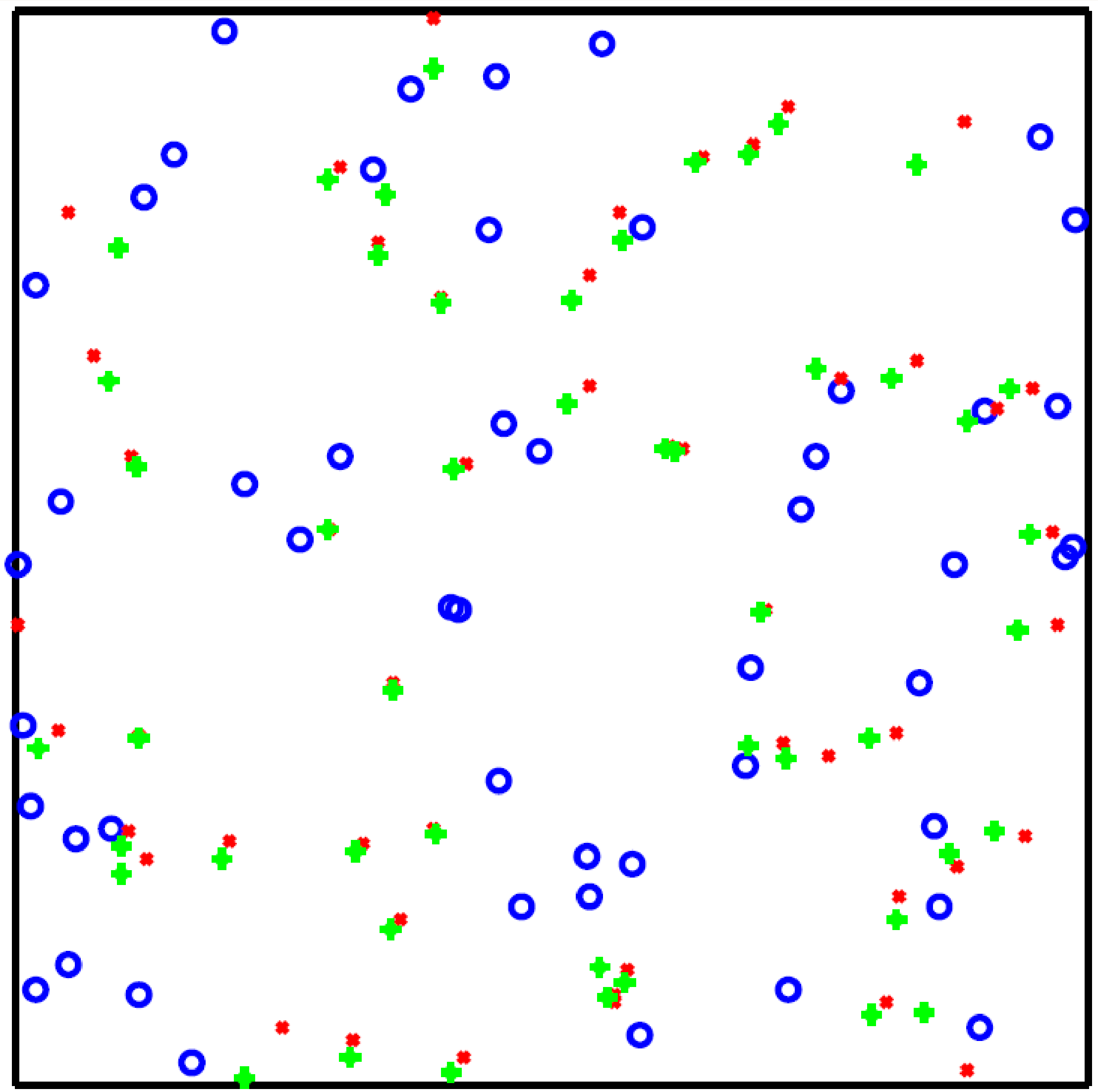
What if no initial configuration of points is known?

Several algorithms exist for nonmetric MDS, but they are all computationally expensive

Landmark nonmetric MDS

- Pick random subset of the data
- Learn an embedding of these “landmark points” via (expensive) nonmetric MDS algorithms
- Use nonmetric triangulation to embed rest of dataset

Landmark nonmetric MDS



$$n_L = 50$$

$$n_T = 50$$

$$|\mathcal{T}| = 200$$

Summary

- Simple convex algorithm for nonmetric triangulation that easily handles
  - noisy observations
  - highly incomplete comparisons
- Natural extension to nonmetric MDS via the method of "landmark points"
- Open questions
  - theoretical analysis
  - active selection of comparisons

Thank You!