A Compressive Introduction to Compressive Sensing

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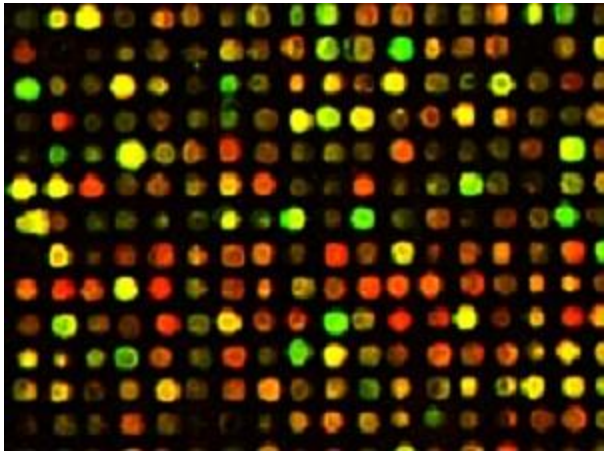
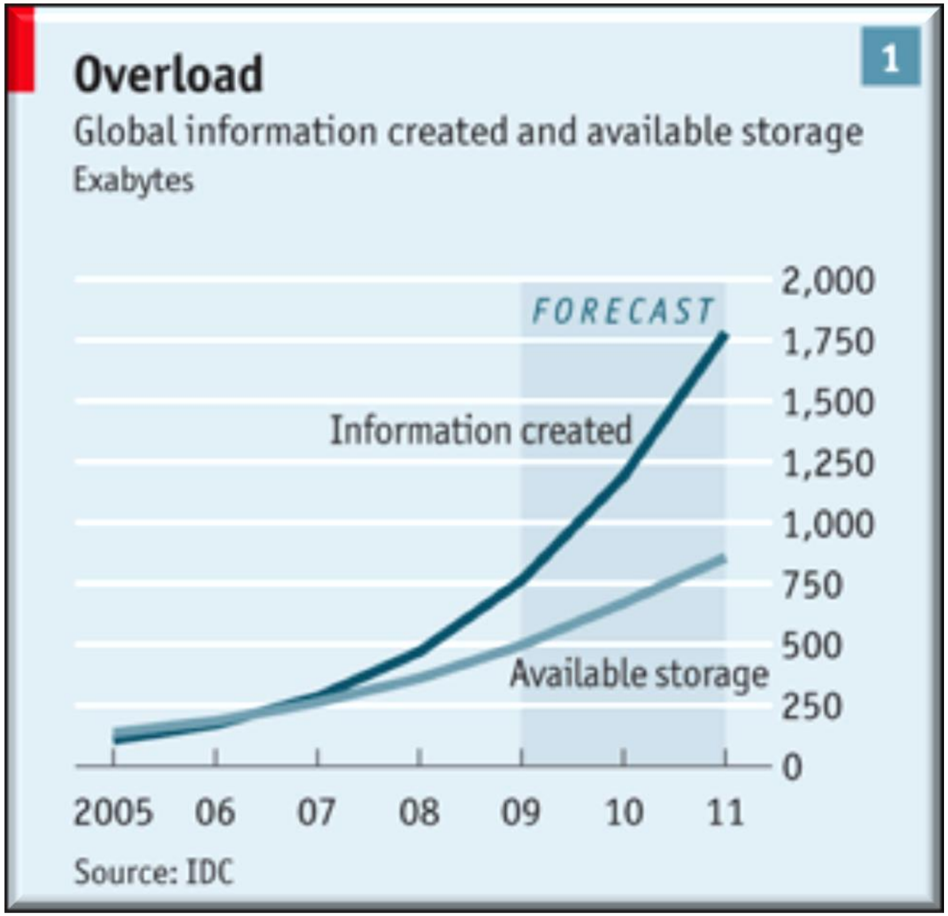
Sensor Explosion



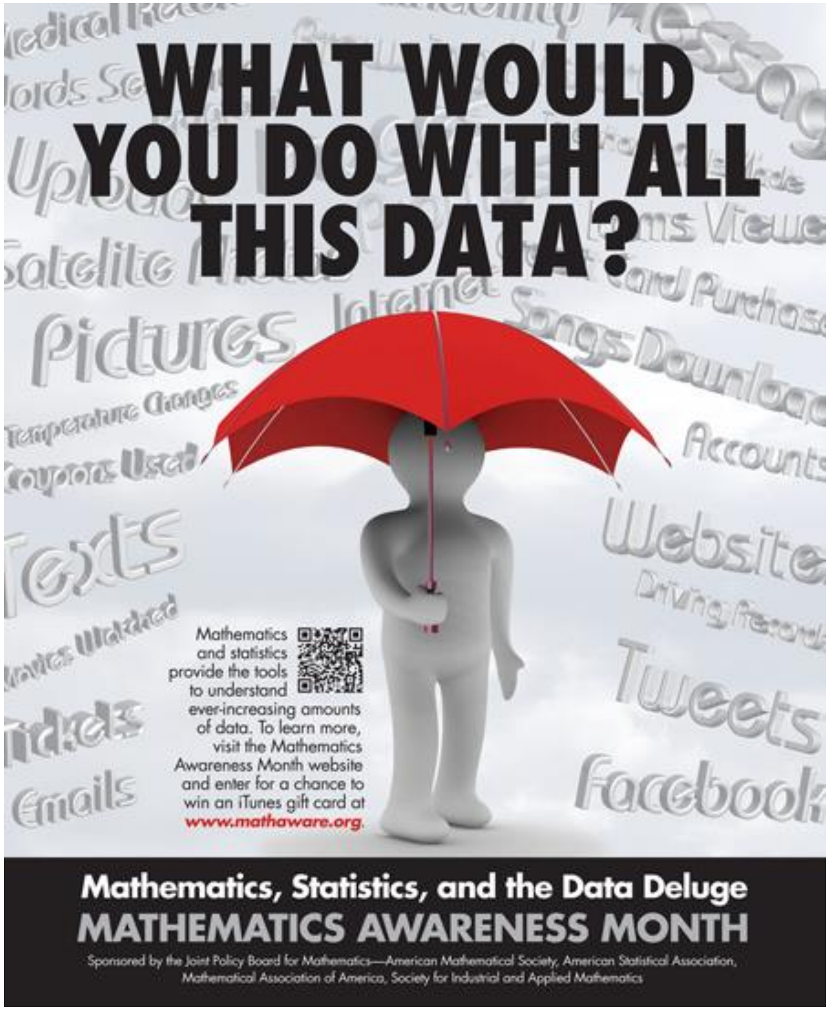
Data Deluge


The Economist
Obama the warrior
Misgoverning Argentina
The economic shift from West to East
Genetically modified crops blossom
The right to eat cats and dogs
The data deluge
AND HOW TO HANDLE IT: A 14-PAGE SPECIAL REPORT

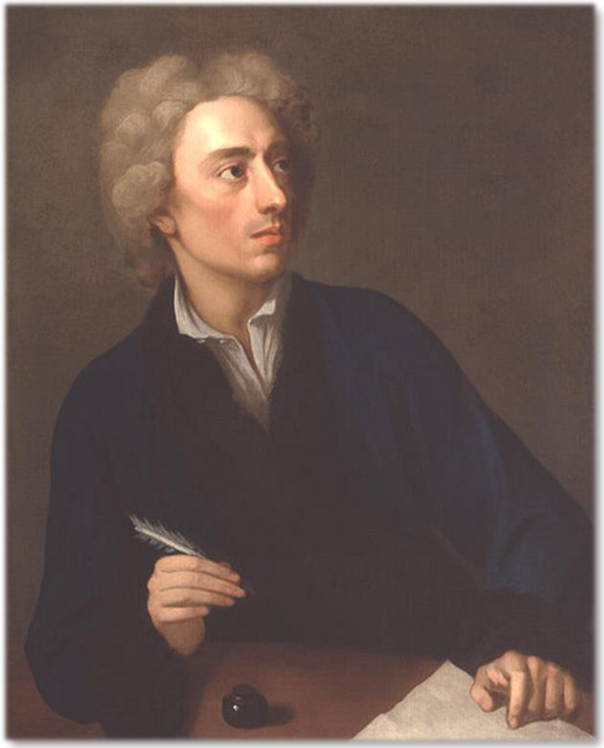

Overload
1
Global information created and available storage
Exabytes
FORECAST
Information created
Available storage
2,000
1,750
1,500
1,250
1,000
750
500
250
0
2005
06
07
08
09
10
11
Source: IDC

2012 Math Awareness Month

Mathematics, Statistics, and the Data Deluge



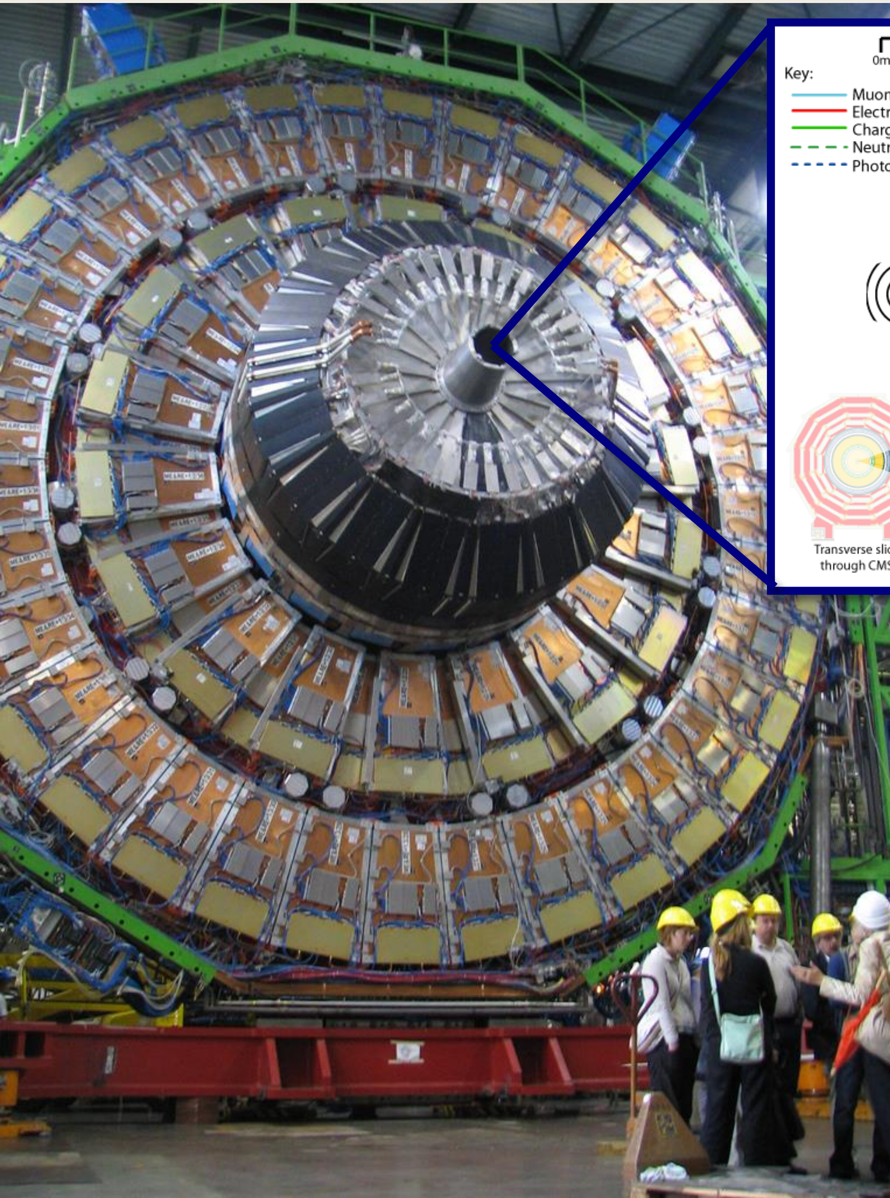
Ye Olde Data Deluge

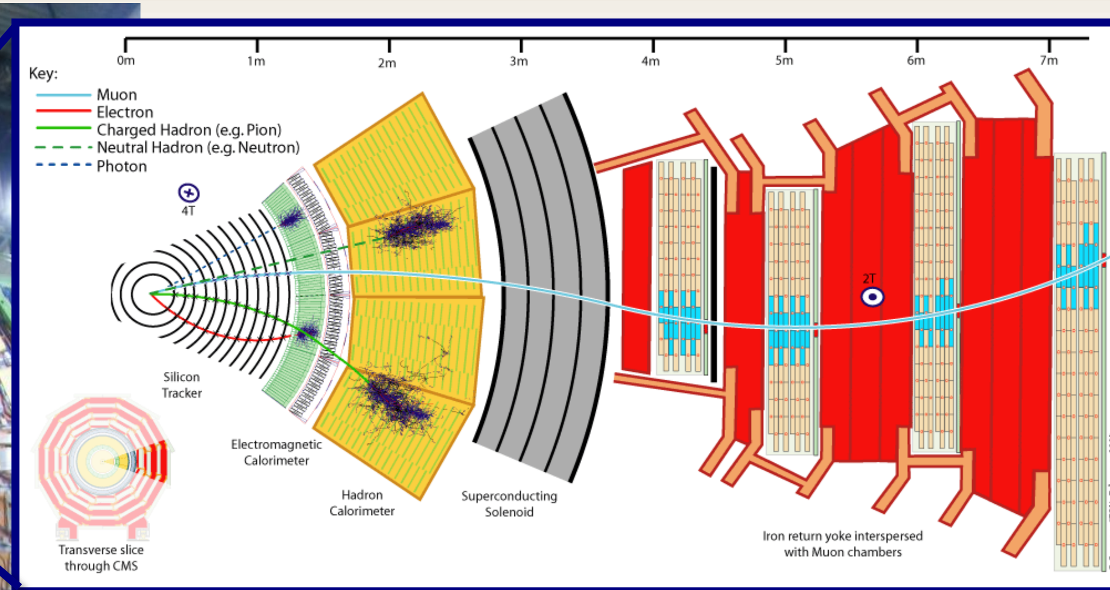


"Paper became so cheap, and printers so numerous, that a deluge of authors covered the land"

Alexander Pope, 1728

Large Hadron Collider at CERN



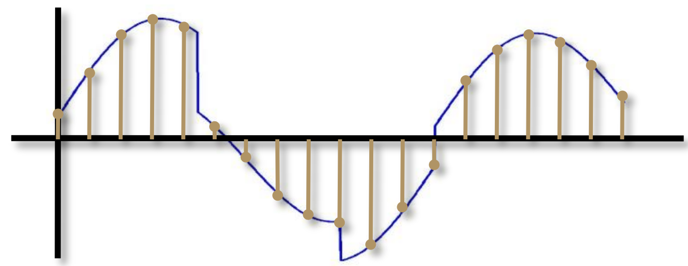


Compact Muon Solenoid detector

320 terabits per second raw data

Stop-gap: perform ad-hoc triage to 800 Gbps, recording only "interesting events"

Digital Revolution



"If we sample a signal at twice its highest frequency, then we can recover it exactly."

Whittaker-Nyquist-Kotelnikov-Shannon

Data, Data Everywhere…

Do we *really* need so many samples?

Most *natural* signals have *simple* characterizations

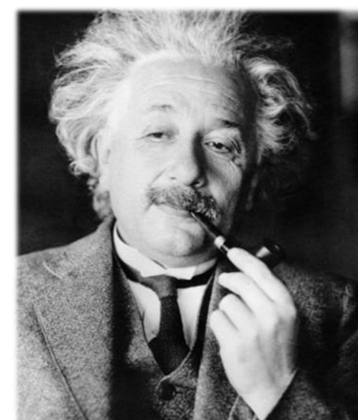
Simplicity Through History

“Entities must not be multiplied unnecessarily”

-William of Occam

“Simplicity is the ultimate sophistication”

-Leonardo da Vinci

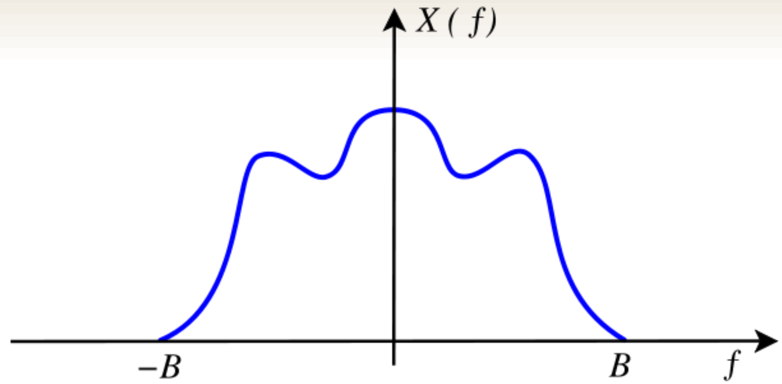


“Make everything as simple as possible, but not simpler”

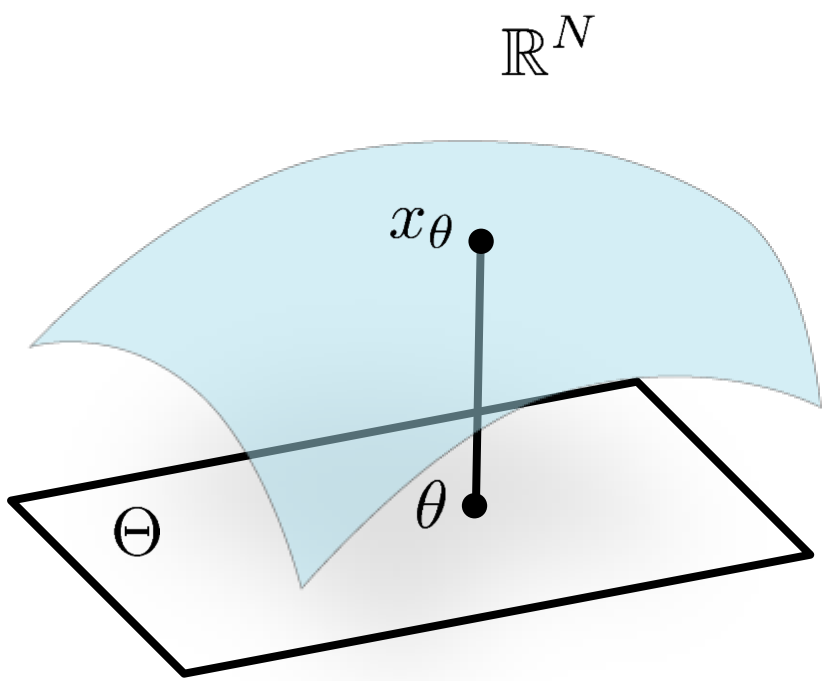
-Albert Einstein

Simple Signals

- Bandlimited signals



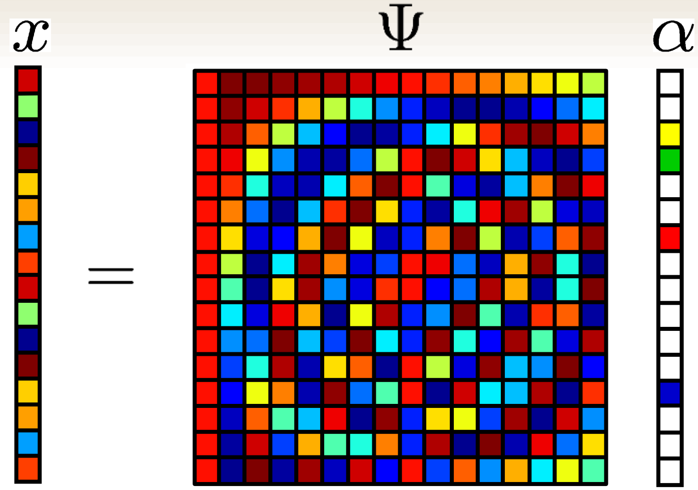
- Parametric signals



- Sparse signals

Sparsity

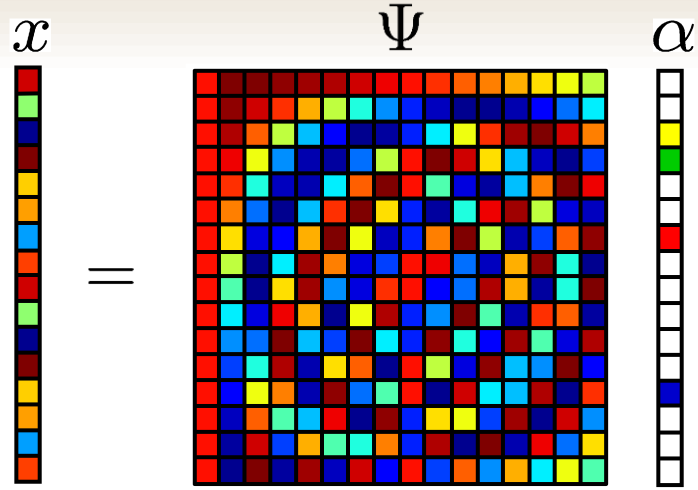
$$x = \sum_{j=1}^{N} \alpha_j \psi_j$$
$$= \Psi \alpha$$



S nonzero entries

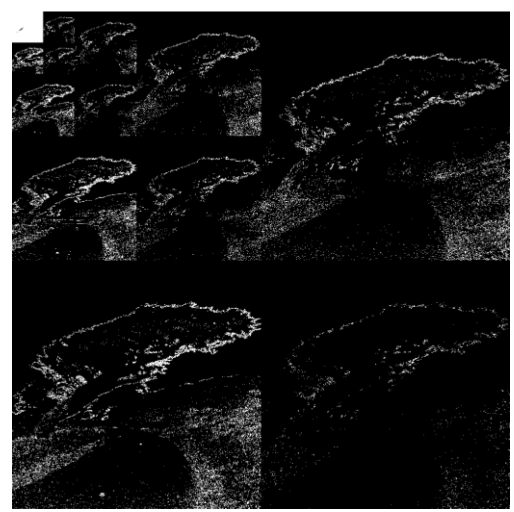
Sparsity

$$x = \sum_{j=1}^{N} \alpha_j \psi_j$$
$$= \Psi \alpha$$



S nonzero entries

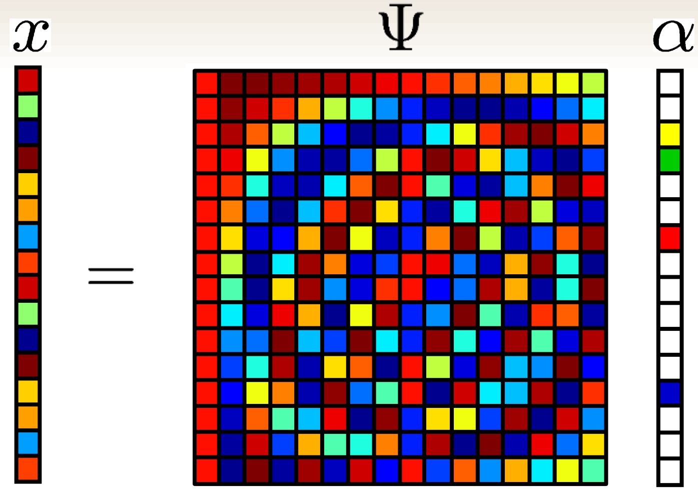
N pixels



$S \ll N$ large wavelet coefficients

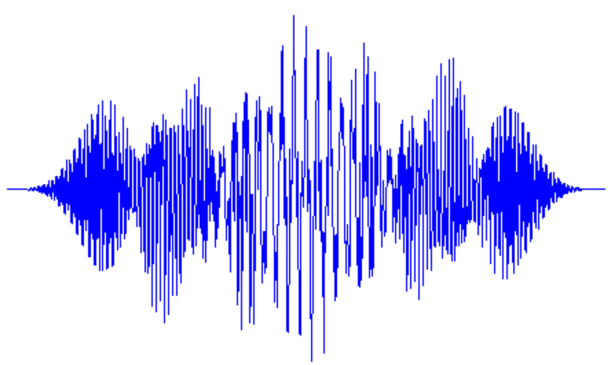
Sparsity

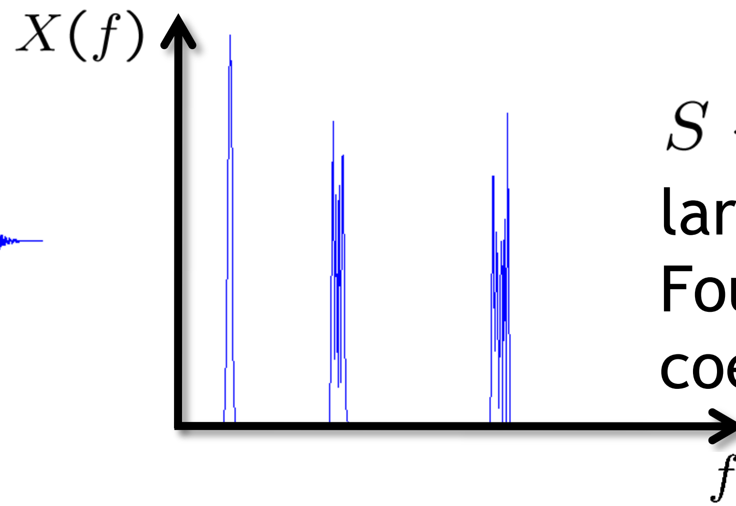
$$x = \sum_{j=1}^{N} \alpha_j \psi_j$$
$$= \Psi \alpha$$



S nonzero entries

N samples



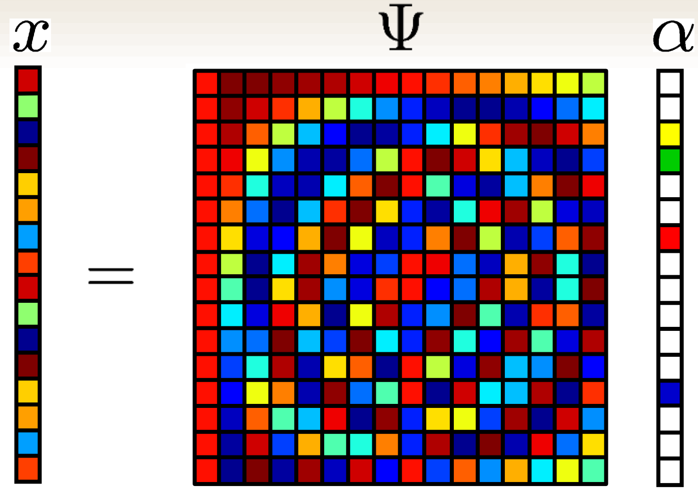


$S \ll N$ large Fourier coefficients

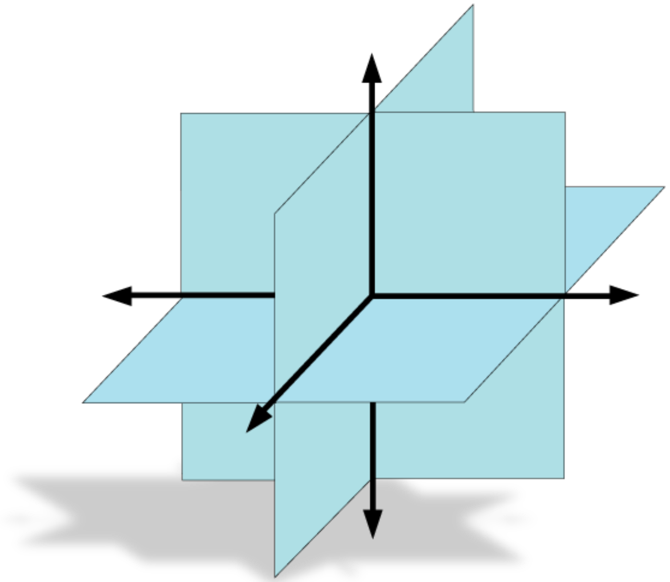
Sparsity

$$x = \sum_{j=1}^{N} \alpha_j \psi_j$$

$$= \Psi \alpha$$



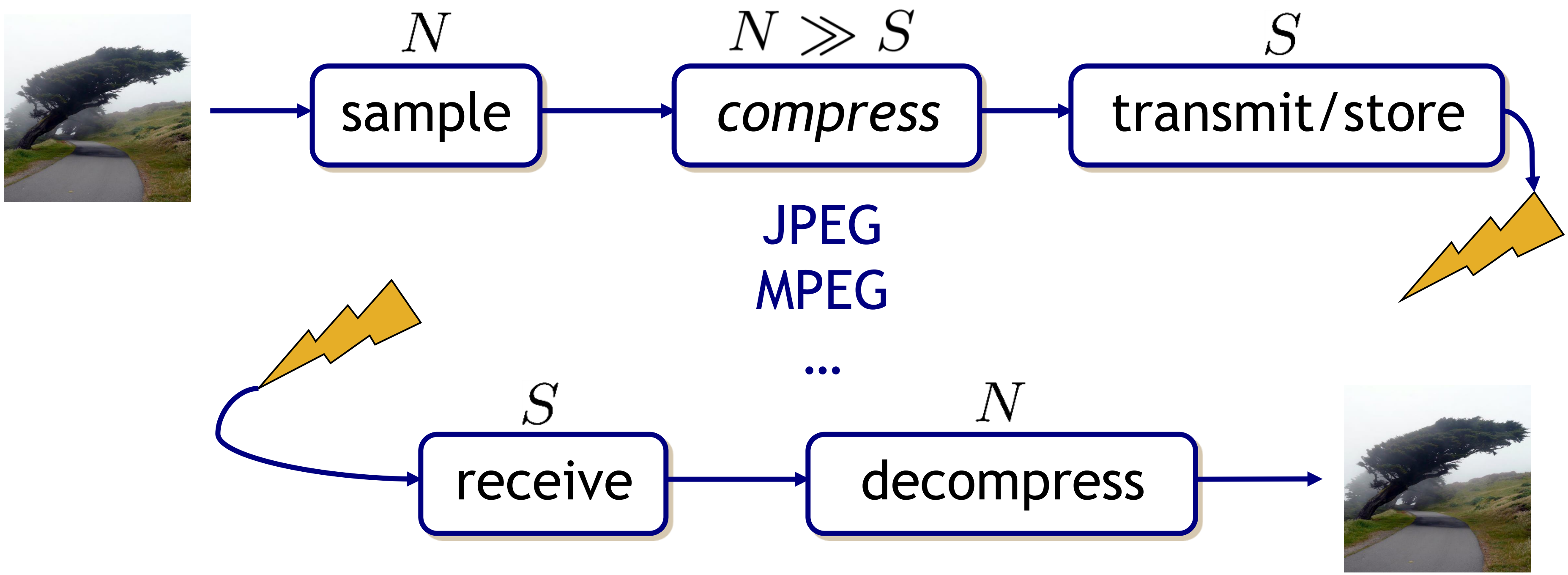
S nonzero entries



Sample-Then-Compress Paradigm

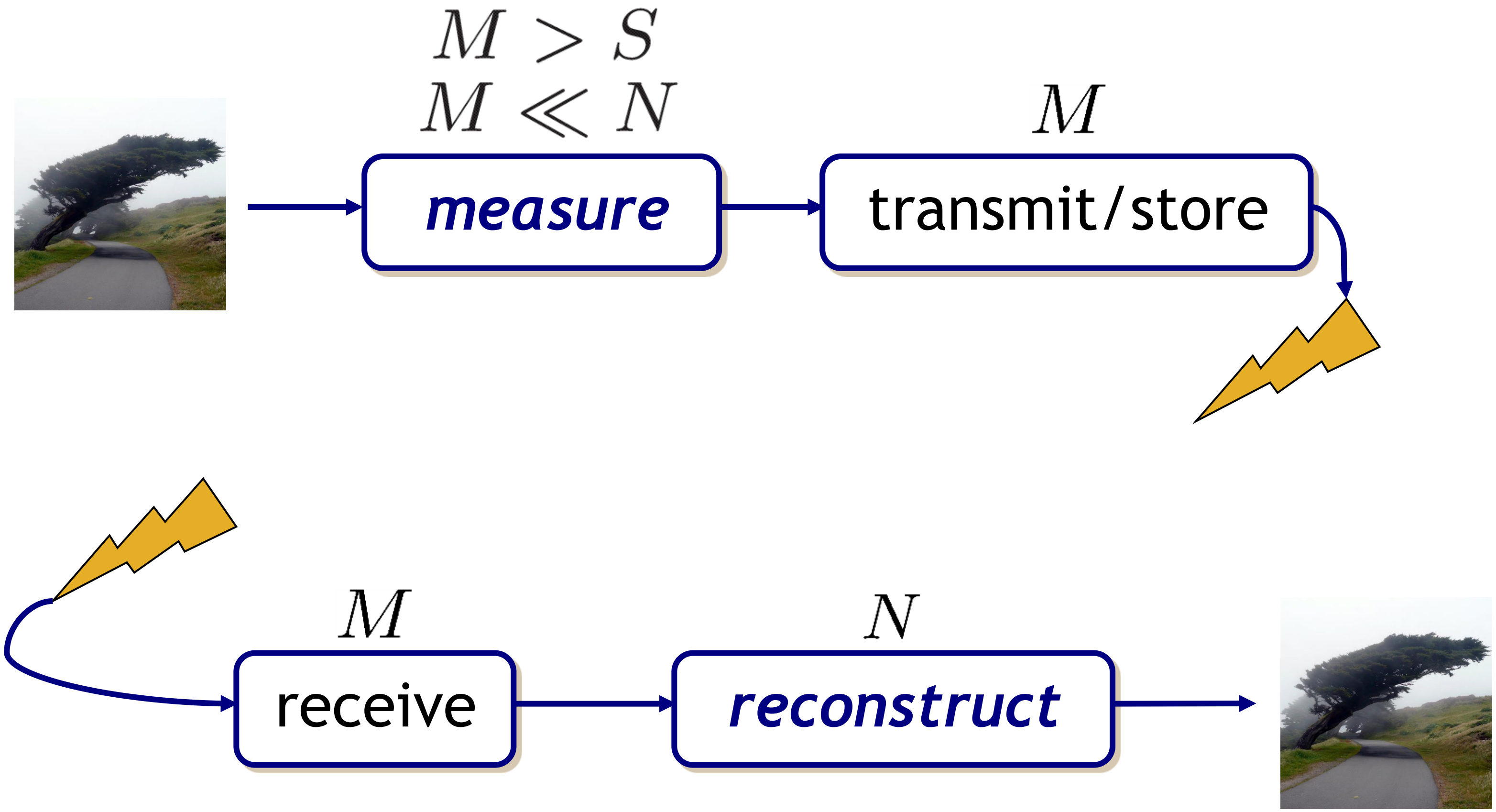
Standard paradigm for digital data acquisition

- *sample* data
- *compress* samples



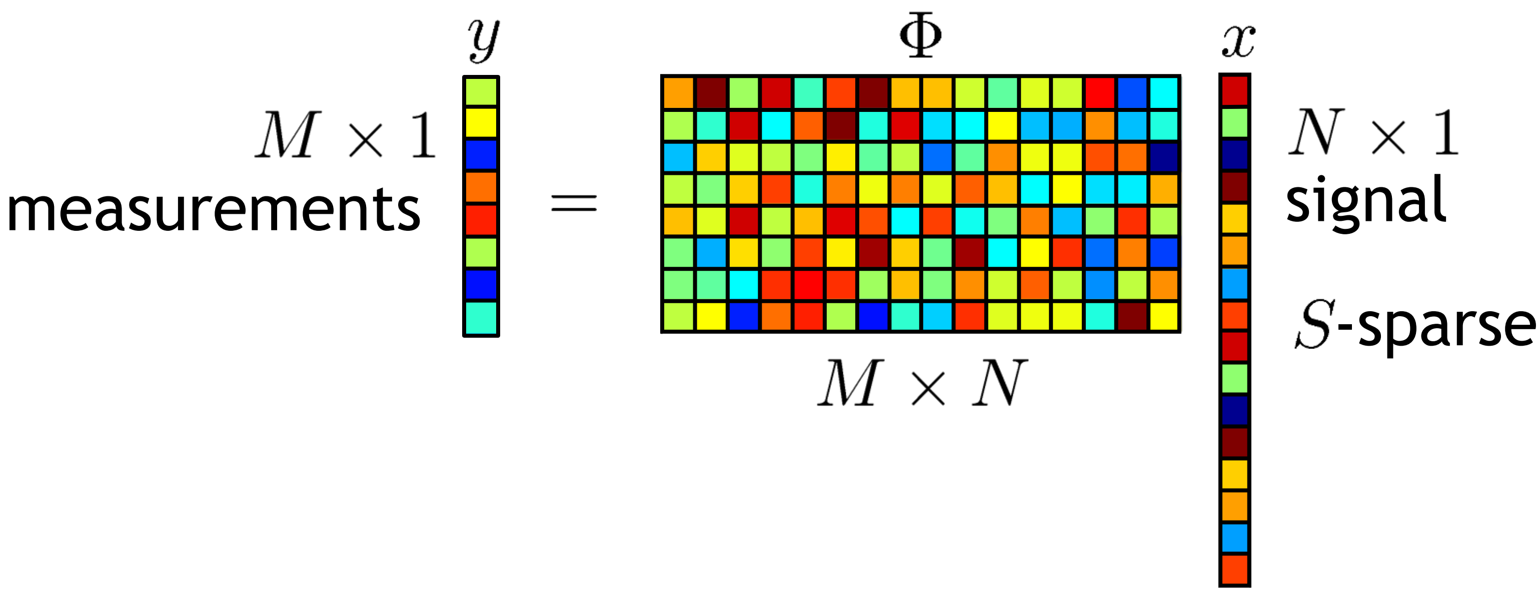
Sample-then-compress paradigm is *wasteful*

Compressive Sensing



Compressive Sensing

Replace samples with general *linear measurements*

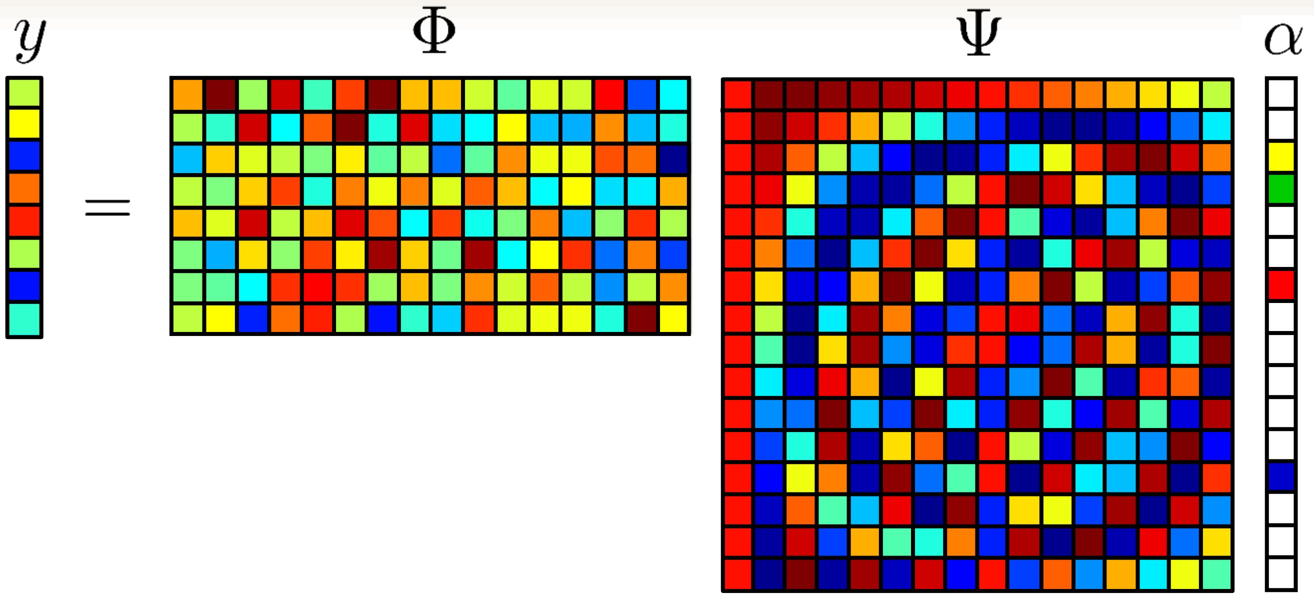


Core challenges:

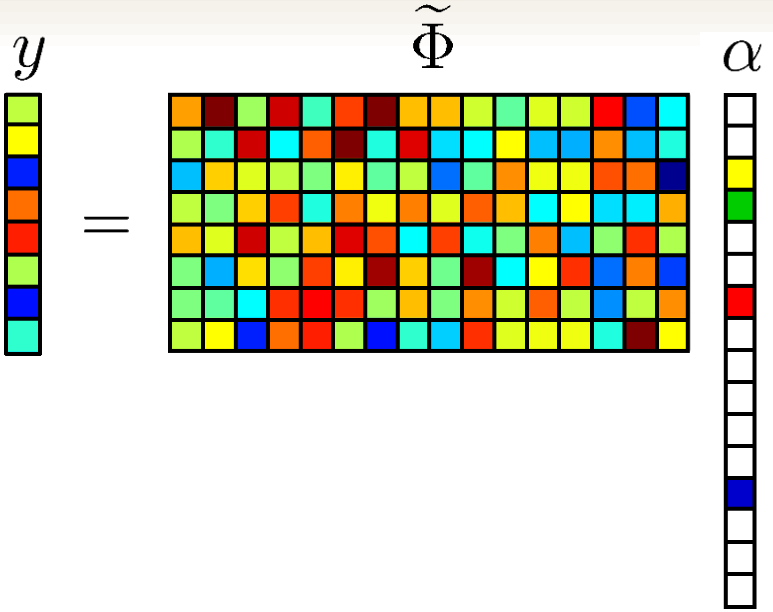
- how to design Φ ?
- how to recover x ?

[Donoho; Candès, Romberg, Tao - 2004]

Sparse Signal Recovery



Sparse Signal Recovery



System of M equations with N unknowns

All but S of the unknowns are zero

Goal: Determine which entries are nonzero, then estimate their values

Sparse Signal Recovery

given $y = \Phi x$
find x

does not lead to sparse solutions → $$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \sum_{j=1}^{N} x_j^2 \quad \text{s.t.} \quad y = \Phi x$$

nonconvex NP-Hard → $$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \sum_{j=1}^{N} \widetilde{x}_j \quad \text{s.t.} \quad y = \Phi x$$

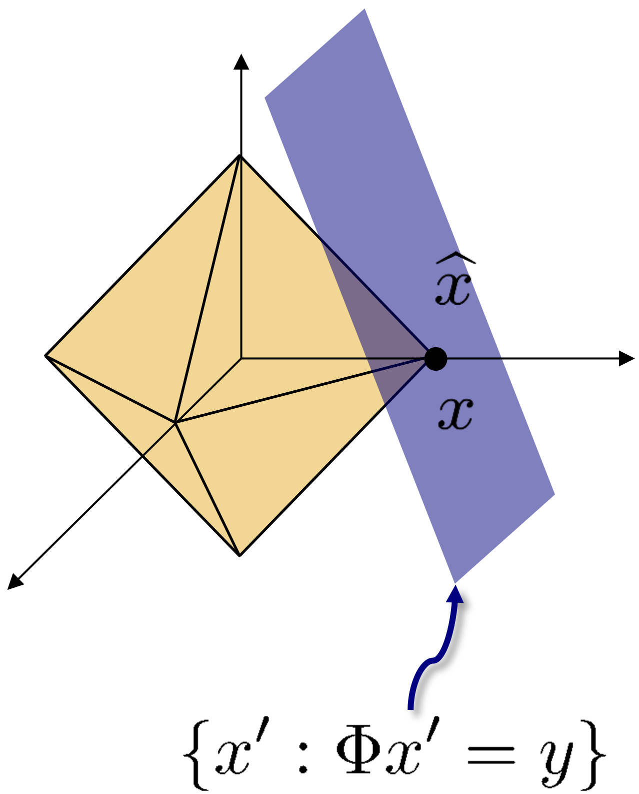
$$\widetilde{x}_j = \begin{cases} 1 & \text{if } x_j \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

convex linear program → $$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \sum_{j=1}^{N} |x_j| \quad \text{s.t.} \quad y = \Phi x$$

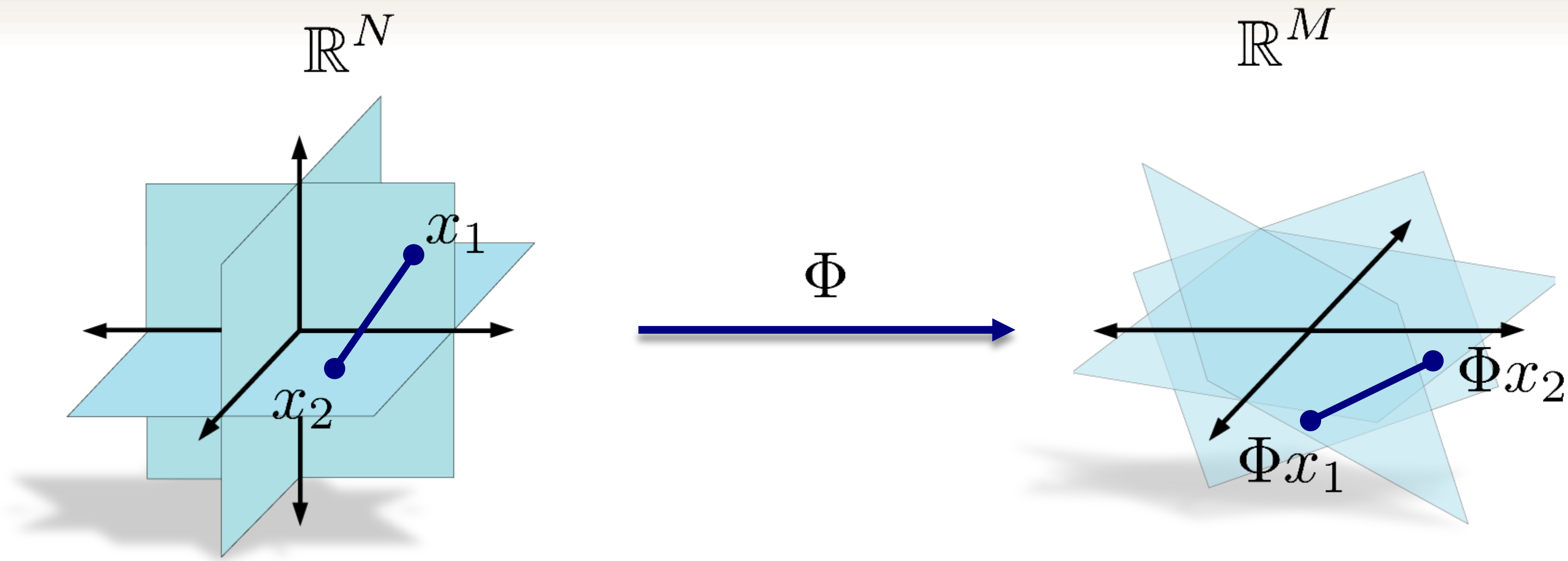
Why ℓ_1-Minimization Might Work

$$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \|x\|_1$$

$$\text{s.t.} \quad y = \Phi x$$



Restricted Isometry Property (RIP)

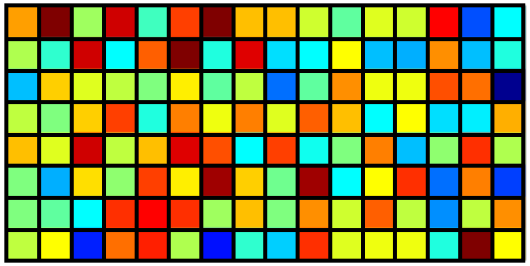


$\|\Phi x_1 - \Phi x_2\|_2^2 \approx \|x_1 - x_2\|_2^2$ for all sparse x_1, x_2

How to Get an RIP Matrix

Choose a *random matrix*

- fill out the entries of Φ with i.i.d. samples from a sub-Gaussian distribution
- project onto a "random subspace"



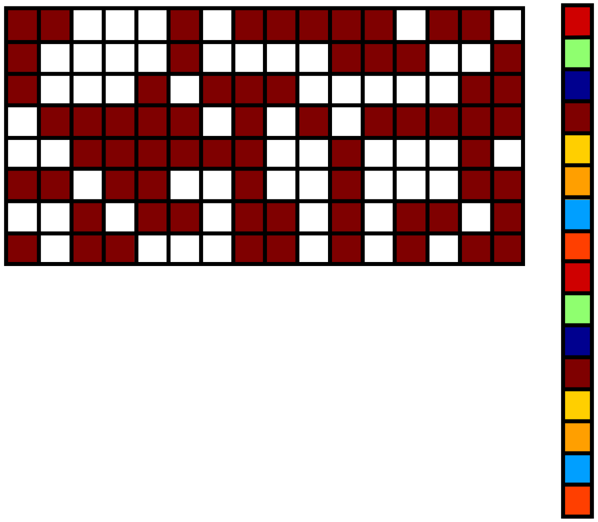
$$M = O(S \log(N/S)) \ll N$$

Many more structured options are now available

Sparse Recovery Guarantees

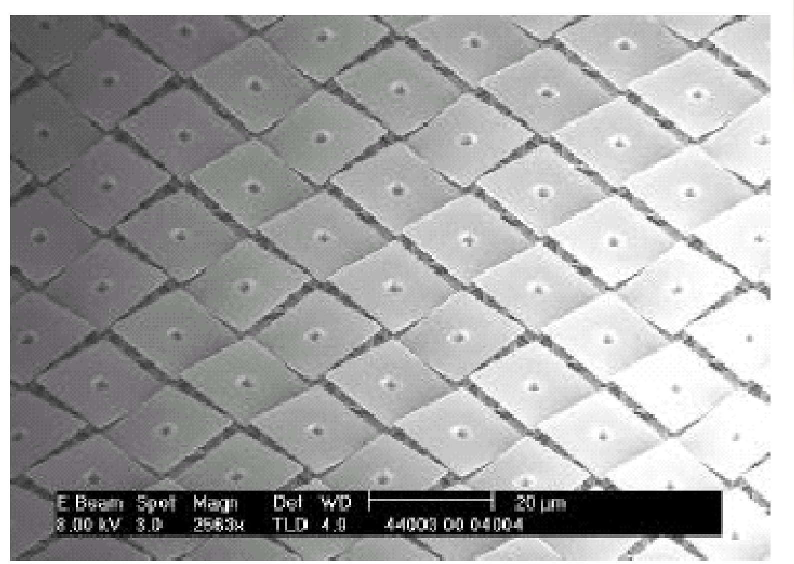
- Optimization / ℓ_1 -minimization
- Greedy algorithms
  - matching pursuit
  - orthogonal matching pursuit (OMP)
  - Stagewise OMP (StOMP), regularized OMP (ROMP)
  - CoSaMP, Subspace Pursuit, IHT, …
- If Φ satisfies the RIP, then any of these algorithms can successfully recover x

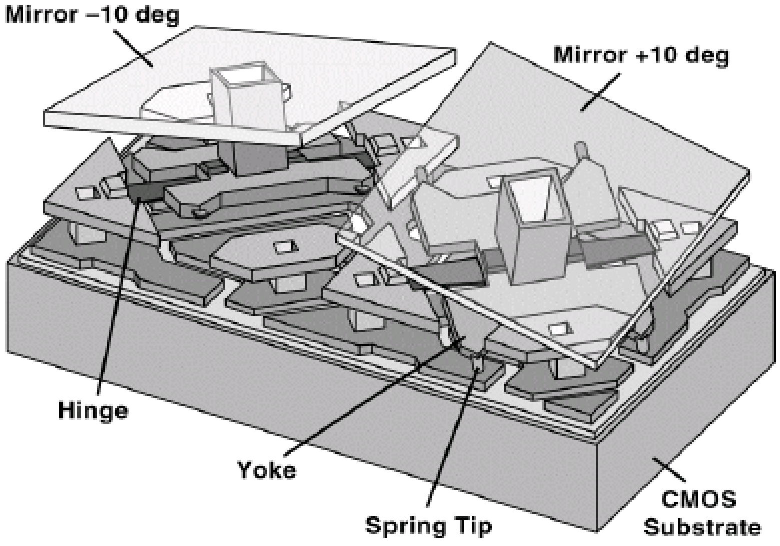
“Single-Pixel Camera”



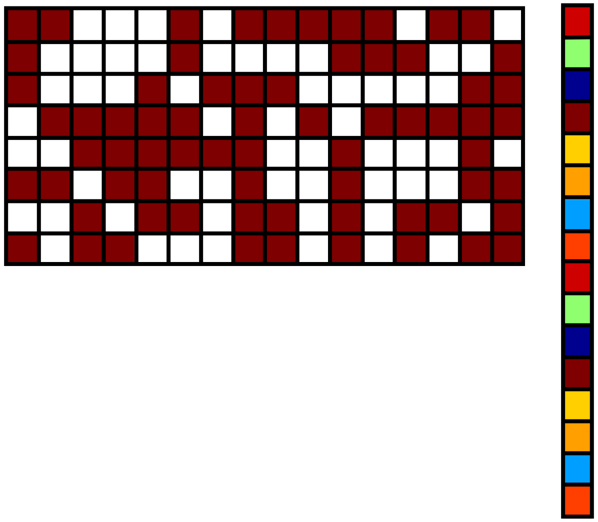
[Duarte, Davenport, Takhar, Laska, Sun, Kelly, Baraniuk - 2008]

TI Digital Micromirror Device





“Single-Pixel Camera”



[Duarte, Davenport, Takhar, Laska, Sun, Kelly, Baraniuk - 2008]

Conclusions

- The theory of compressive sensing allows for new sensor designs, but requires new techniques for signal recovery

- Underdetermined systems of equations with sparse solutions arise in many other contexts

- “Simplicity” has many incarnations
  - sparsity
  - structured sparsity
  - finite rate of innovation, manifold, parametric models
  - low-rank matrices