Compressive Sensing

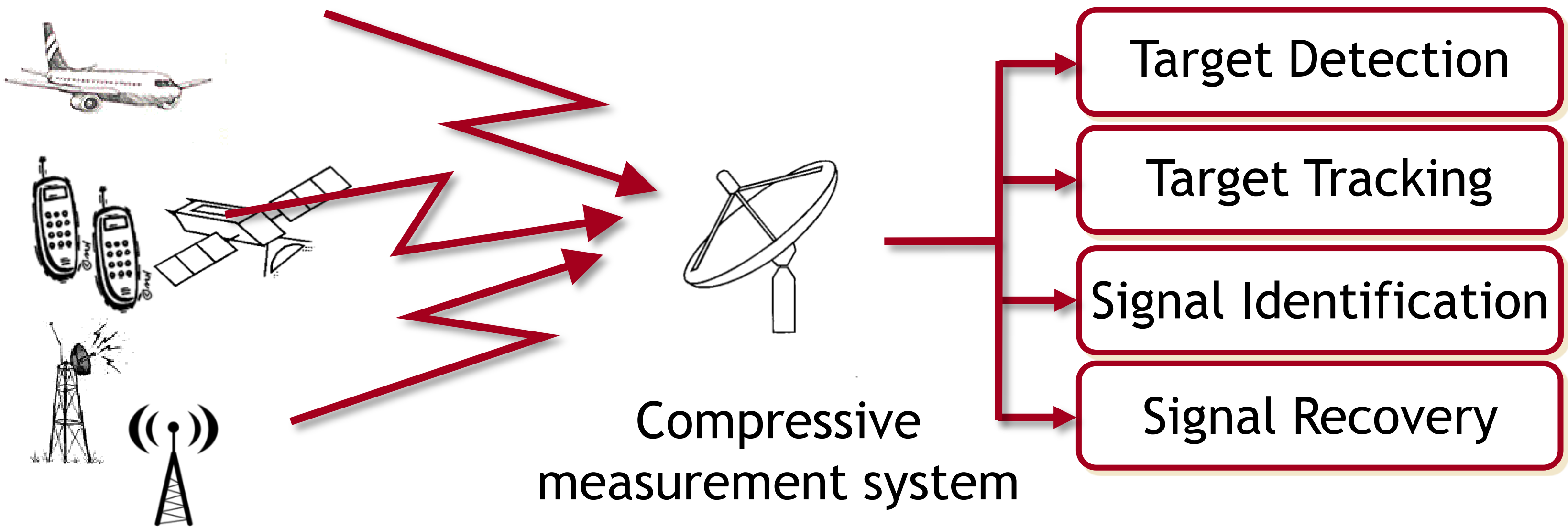
Part V: Beyond Recovery

Mark A. Davenport

Stanford University
Department of Statistics

Compressive Signal Processing

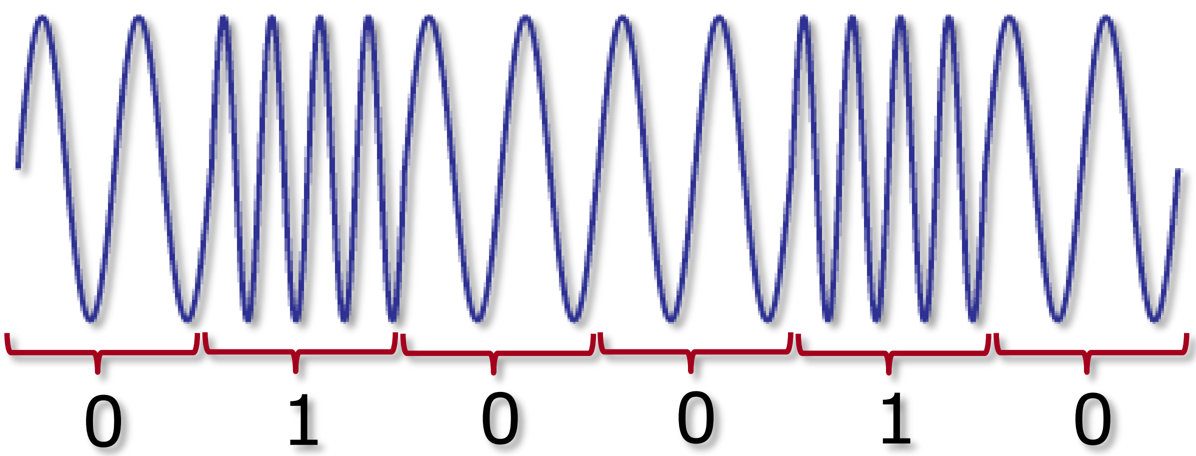
Random measurements are ***information scalable***



When and how can we directly solve signal processing problems directly from compressive measurements?

Example: FM Signals

- Can we directly recover a *baseband voice signal* without recovering the modulated waveform?
- Suppose we have compressive measurements of a digital communication signal (FSK modulated)



- Can we directly recover the encoded *bitstream* without first recovering the measured waveform?

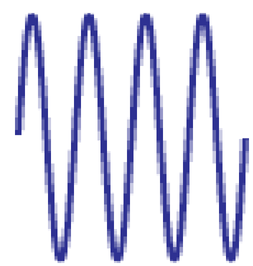
Compressive Likelihood-Ratio Test

- Suppose that we are synchronized and know the exact carrier frequencies of the signal

- Window measurements according bit-intervals

s_0

s_1



$$\mathcal{H}_0 : x = s_0 + n \qquad \mathcal{H}_0 : y = \Phi(s_0 + n)$$

$$\mathcal{H}_1 : x = s_1 + n \qquad \mathcal{H}_1 : y = \Phi(s_1 + n)$$

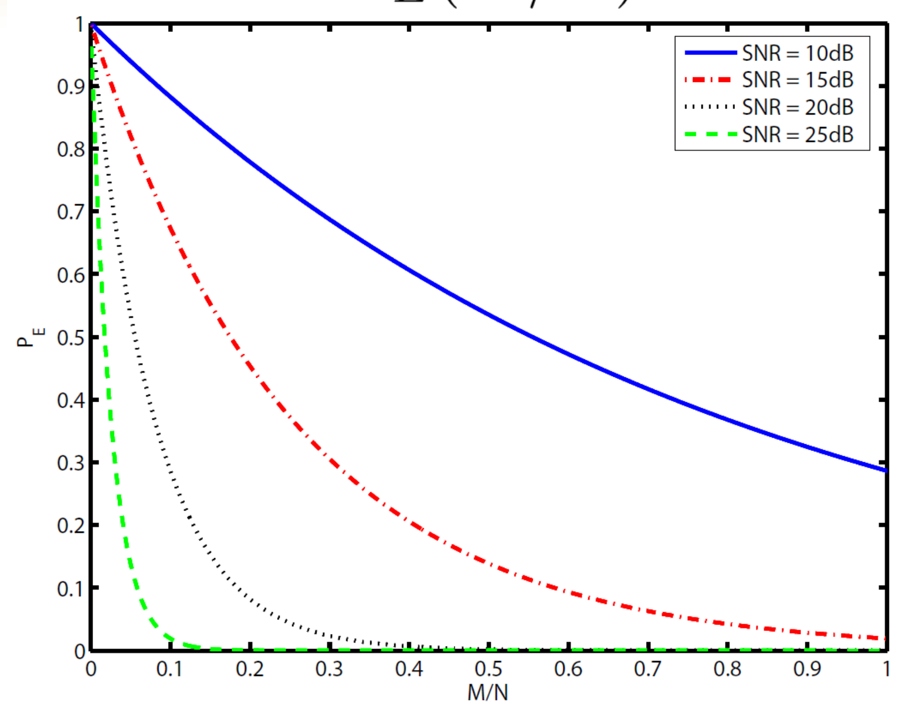
Compressive Classification

- Matched filter
  (in colored noise)

$$t_0 = s_0^T \Phi^T (\Phi\Phi^T)^{-1} y$$

$$t_1 = s_1^T \Phi^T (\Phi\Phi^T)^{-1} y$$

$P_E(M/N)$



- Expected performance given by

$$P_D(\alpha) = Q\left(Q^{-1}(\alpha) - \frac{(s_0 - s_1)^T \Phi^T (\Phi\Phi^T)^{-1} \Phi (s_0 - s_1)}{\sigma}\right)$$

$$\approx Q\left(Q^{-1}(\alpha) - \sqrt{\frac{M}{N}} \frac{\|s_0 - s_1\|_2}{\sigma}\right)$$

[D, Boufounos, Wakin, and Baraniuk – 2010]

Compressive Detection

$$\mathcal{H}_0 : x = n \qquad\qquad \mathcal{H}_0 : y = \Phi n$$

$$\mathcal{H}_1 : x = s + n \qquad\qquad \mathcal{H}_1 : y = \Phi(s + n)$$

$$\langle x, s \rangle \gtrless \gamma \qquad\qquad s^T \Phi^T (\Phi \Phi^T)^{-1} y \gtrless \widetilde{\gamma}$$

- If Φ is an orthoprojector, then

$$s^T \Phi^T (\Phi \Phi^T)^{-1} y = \sqrt{\frac{N}{M}} s^T \Phi^T y = \sqrt{\frac{N}{M}} \langle y, \Phi s \rangle$$

- ROC given by

$$P_D(\alpha) = Q\left(Q^{-1}(\alpha) - \frac{\|\Phi s\|_2}{\sigma} \right) \approx Q\left(Q^{-1}(\alpha) - \sqrt{\frac{M}{N}} \frac{\|s\|_2}{\sigma} \right)$$

[D, Boufounos, Wakin, and Baraniuk – 2010]

Compressive Estimation

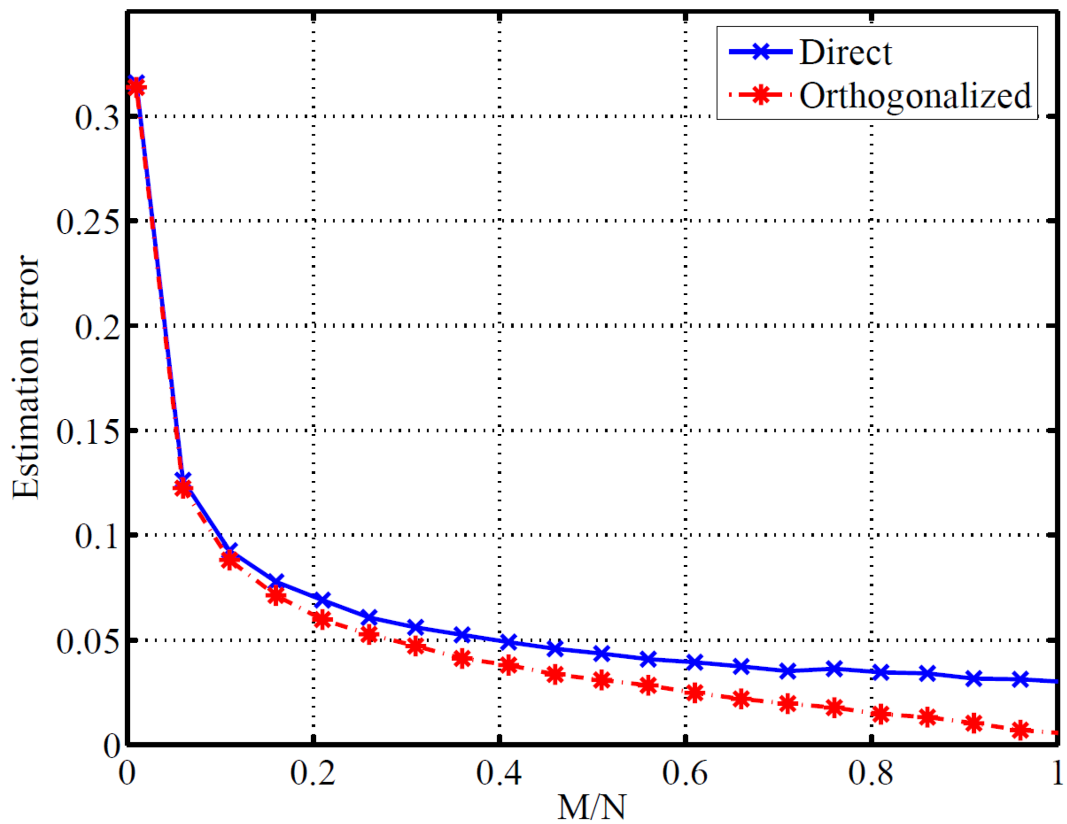
- Suppose we wish to estimate $\langle x, \ell \rangle$ from $y = \Phi(x+n)$.

- Direct: $\langle y, \Phi\ell \rangle$

$$\boxed{\frac{|\langle y, \Phi\ell\rangle - \langle y, \ell\rangle|}{\|x\|_2 \|\ell\|_2} \leq \delta}$$

- Orthogonalized:

$$\ell^T \Phi^T (\Phi\Phi^T)^{-1} y$$



[D, Boufounos, Wakin, and Baraniuk – 2010]

Extreme Sparsity

- In all these examples, we essentially have $S = 1$.

- All the information we need to solve our problem lies in the 1-dimensional subspace spanned by
  - s in the case of detection
  - $s_0 - s_1$ in the case of classification
  - ℓ in the case of estimation

- We have to make a lot of assumptions for these models to be relevant...

- Can these ideas be generalized?

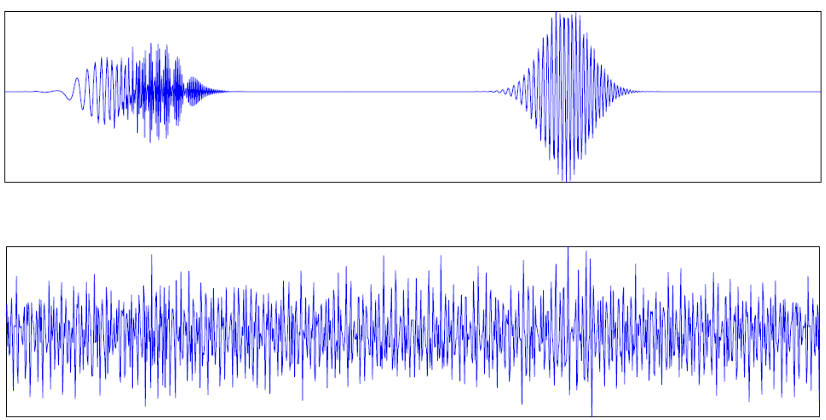
Sparse Signal Classification

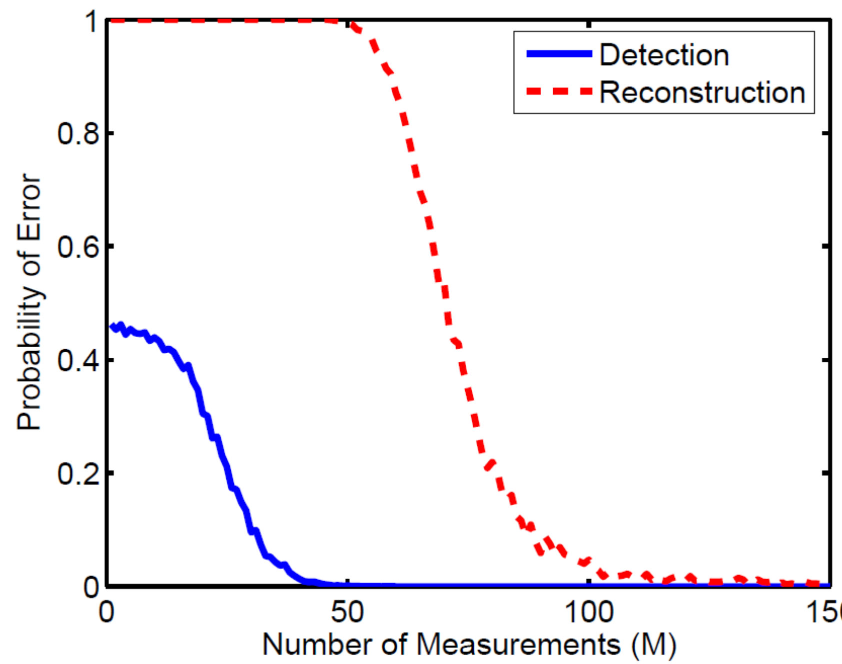
- Suppose

$$\mathcal{H}_0 : y = \Phi\Psi_0\alpha$$
$$\mathcal{H}_1 : y = \Phi\Psi_1\alpha$$
$$\vdots$$
$$\mathcal{H}_P : y = \Phi\Psi_P\alpha$$

unknown but sparse

- Incoherent Detection and Estimation Algorithm(IDEA)





[Duarte, D, Wakin, and Baraniuk – 2006]

Matched Filters

- We may know what signals we are looking for, but we may not know ***where*** to look

$$H_j : x = s_j(t - \theta_j) + n$$

- Elegant solution:

Matched Filter

Compute $\langle x, s_j(t - \theta_j) \rangle$ for all θ_j $\Longleftrightarrow$ $x * s_j(-t)$

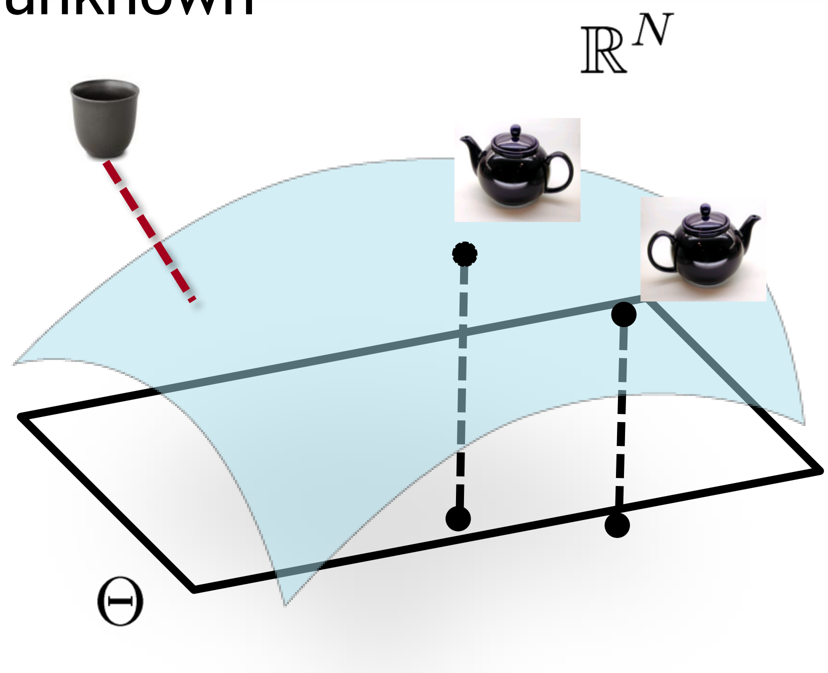
Challenge: Modify the compressive LRT to accommodate ***unknown parameters***

Matched Filter Geometry

- Detection/classification with S unknown articulation parameters

- Images are points in $\mathbb{R}^N$

- As template articulation parameters change, points trace out an S-dimensional manifold

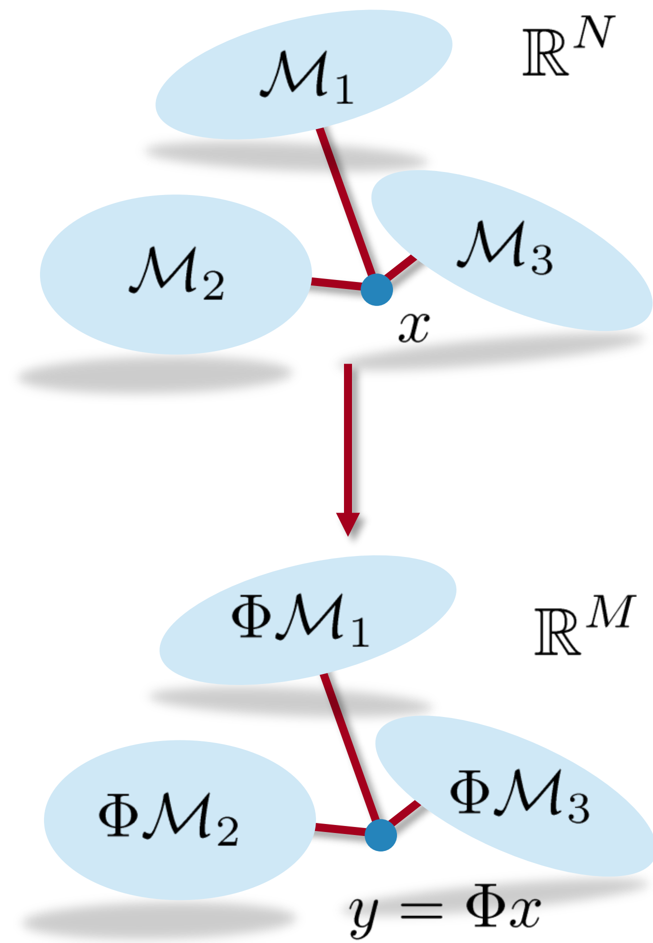


- Classify by finding closest target template to data

- Matched filter based on generalized likelihood ratio test = *closest manifold search*

The *Smashed Filter*

- *Compressive* manifold classification with GLRT
  - nearest-manifold classifier
  - manifolds classified are now $\Phi \mathcal{M}_j$



- To stably embed K manifolds
  - dimension S
  - condition number $1/\tau$
  - volume V

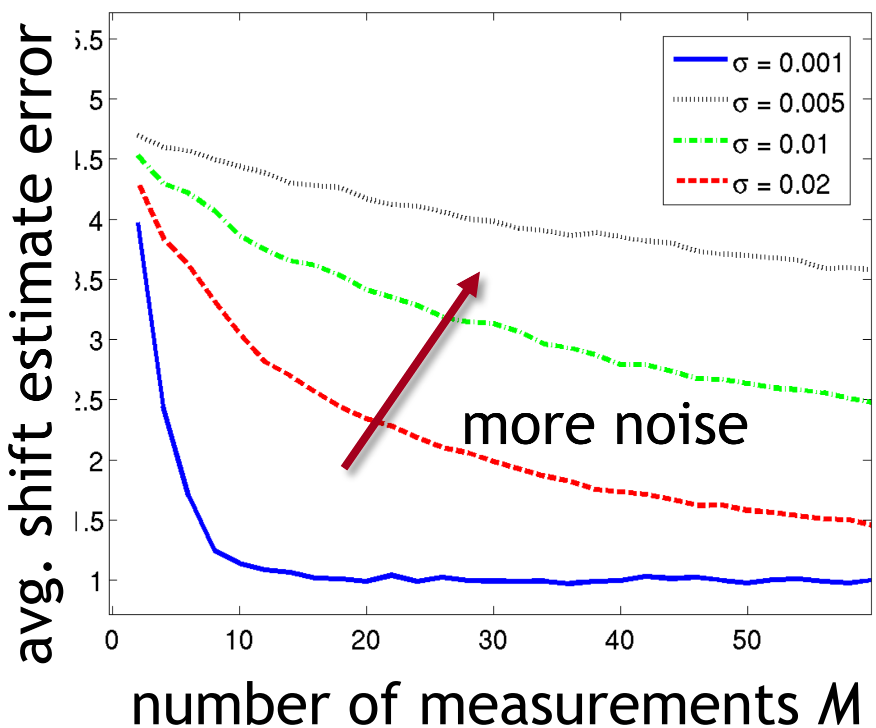
$$M = O(KS \log(NV/\tau))$$

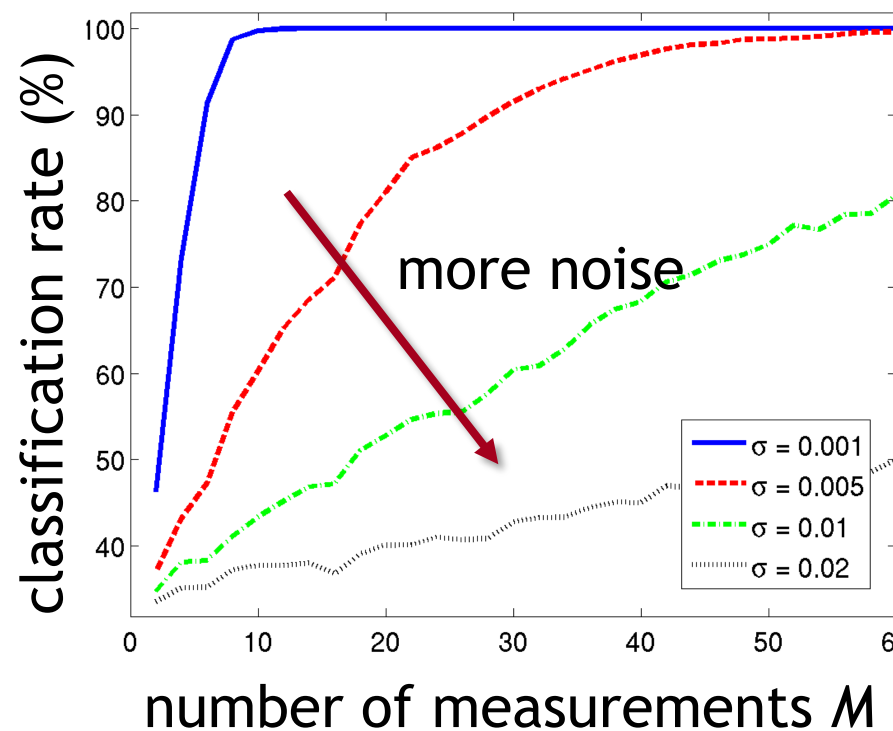
Smashed Filter - Experiments

- 3 image classes
  - T-72 tank
  - schoolbus
  - SUV
- Imaged using single-pixel camera with
  - unknown shift
  - unknown rotation

Smashed Filter

- Random shift and rotation ($S = 3$ dim. manifold)
- AWG noise added to measurements
- Goals: identify most likely shift/rotation parameters
  identify most likely class





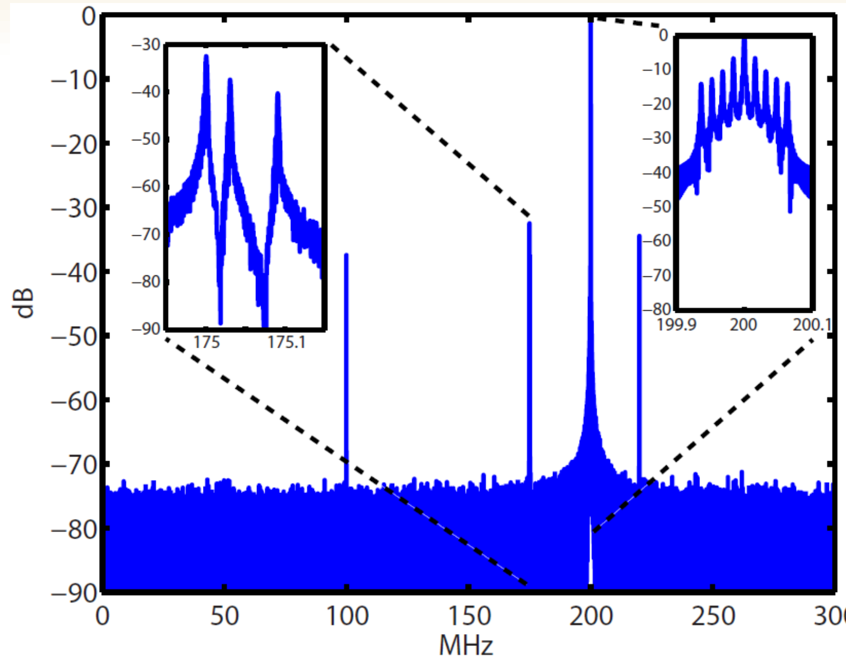
[D, Duarte, Wakin, Laska, Takhar, Kelly, and Baraniuk - 2007]

Putting It All Together

Compressive Radio Receivers

Example Scenario

- 300 MHz bandwidth
- 5 FM signals (12 kHz)
- TV station interference
- Acquire compressive measurements at 30 MHz (20 x undersampled)



We must simultaneously solve several problems

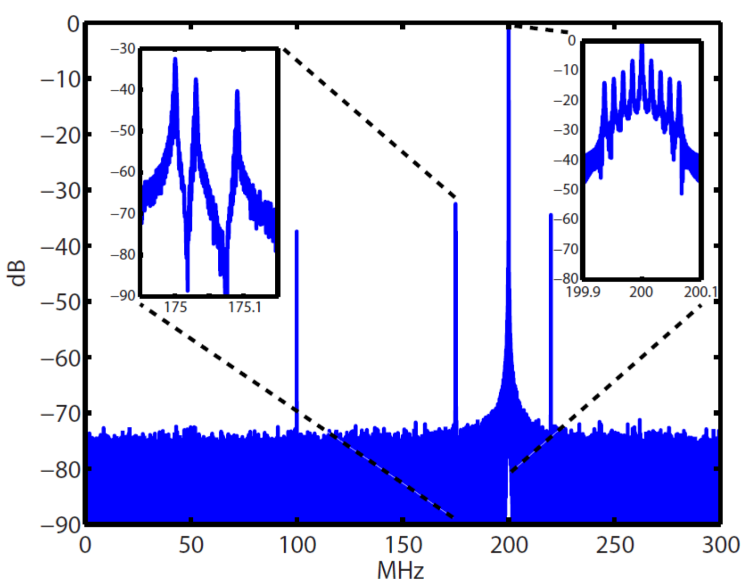
y → detect signal energy → cancel known interferers → filter signals of interest → demod → baseband signals or bitstreams

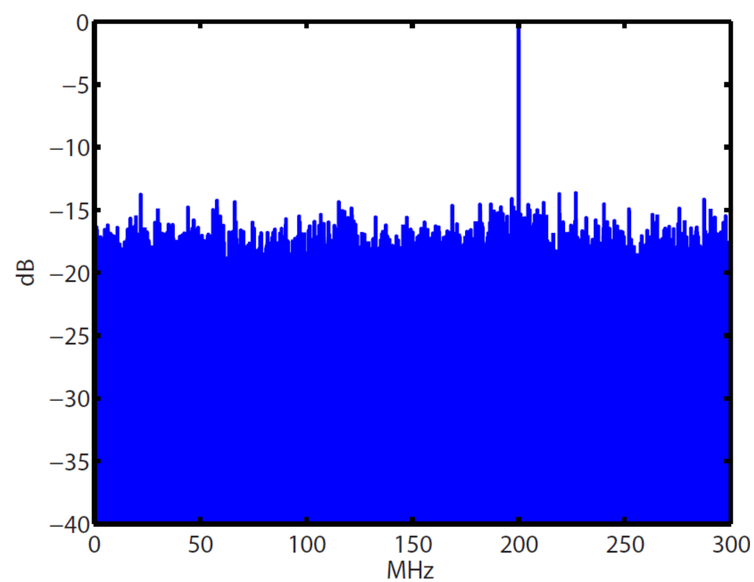
Energy Detection

We need to identify where in frequency the important signals are located

Comprressive Estimation: correlate with projected tones

$$\widehat{F}(k) = |\langle \Phi \cos(2\pi f_k t), y \rangle|$$





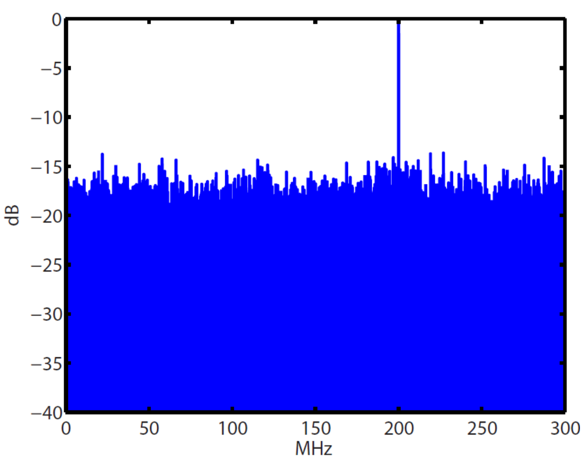
[D, Schnelle, Slavinsky, Baraniuk, Wakin, and Boufounos – 2010]

Filtering

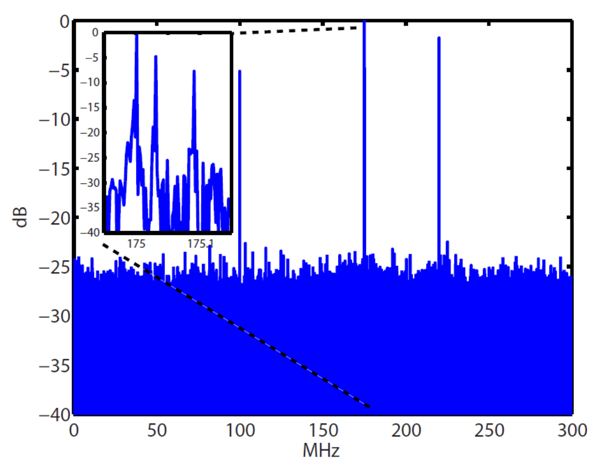
If we have multiple signals, must be able to filter to isolate and cancel interference

$$P = I - \Phi S(\Phi S)^{\dagger}$$

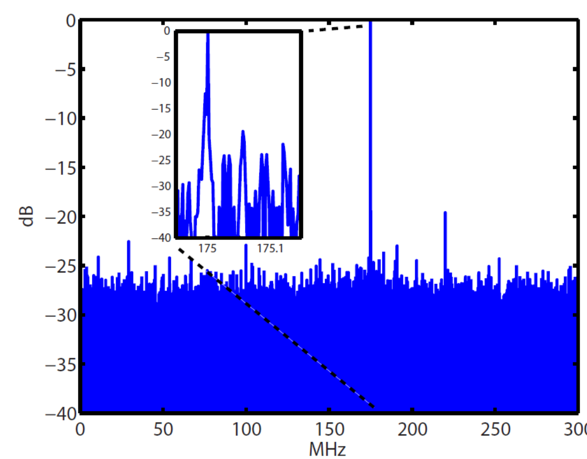
S: Discrete prolate spheroidal sequences



original



after interference cancellation

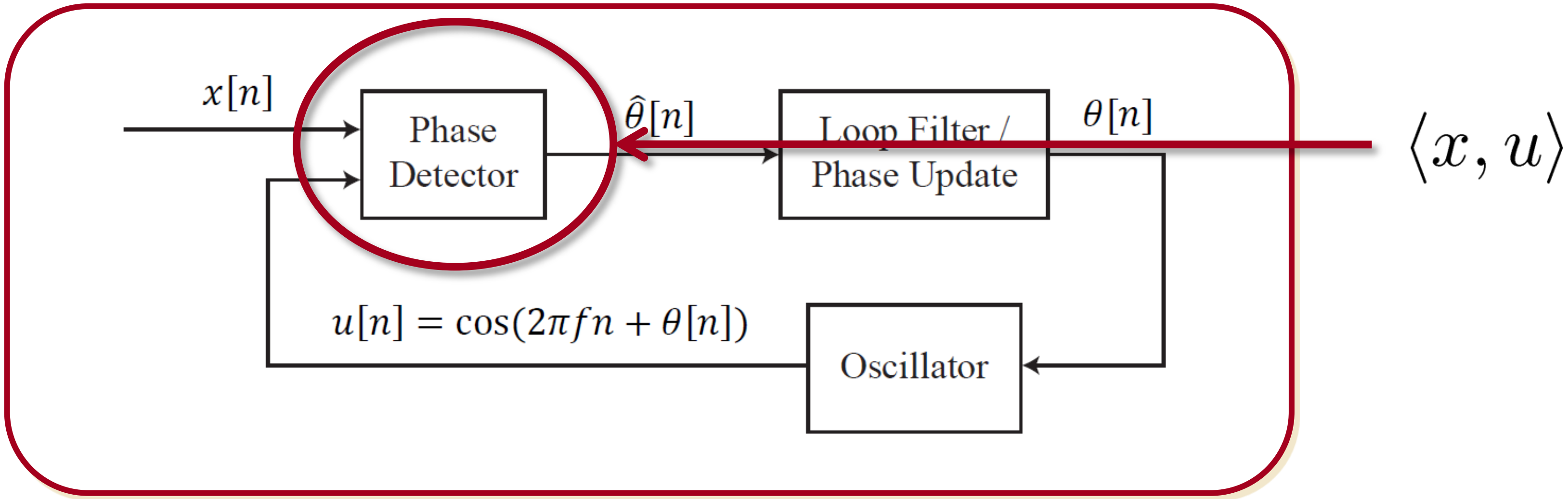


after isolation filtering

[D, Schnelle, Slavinsky, Baraniuk, Wakin, and Boufounos – 2010]

Unsynchronized Demodulation

We can use a phase-locked-loop (PLL) to track deviations in frequency by directly operating on compressive measurements



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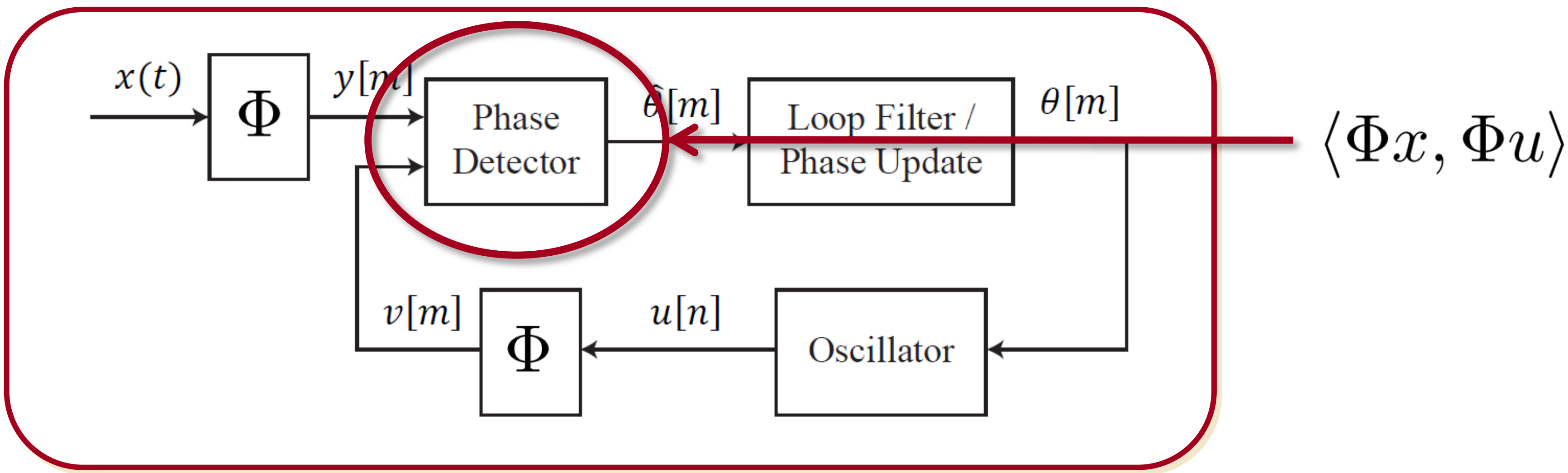
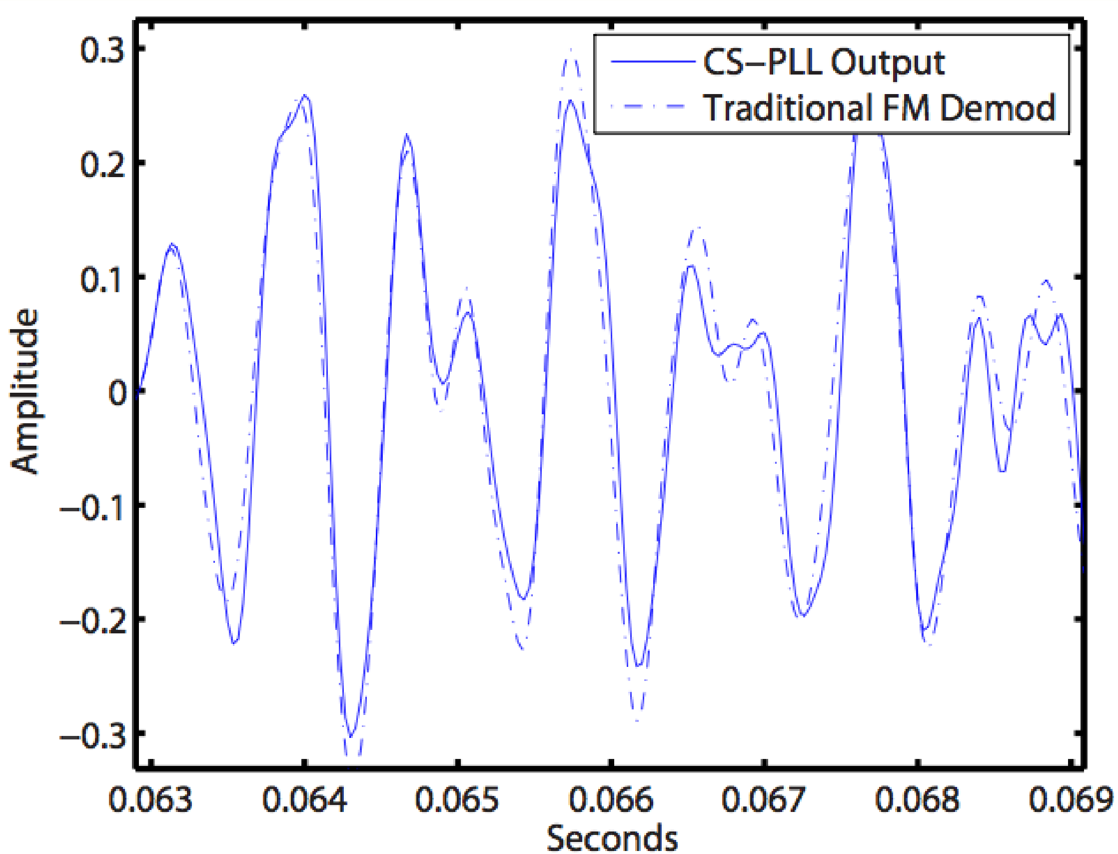


We can directly demodulate signals from compressive measurements *without recovery*

[D, Schnelle, Slavinsky, Baraniuk, Wakin, and Boufounos – 2010]

Compressive Domain Demodulation



CS-PLL with
20x undersampling

[D, Schnelle, Slavinsky, Baraniuk, Wakin, and Boufounos – 2010]

Summary

- **Compressive signal processing**
  - integrates sensing, compression, processing
  - exploits signal sparsity/compressibility
  - enables new sensing modalities, architectures, systems
  - exploits randomness at many levels

- Why CSP works: preserves information in signals with concise geometric structure

  sparse signals | manifolds | low-dimensional models

- **Information scalability** for compressive inference
  - compressive measurements ~ sufficient statistics
  - much less computation required than for recovery